Ph.D. Course on *Vorticity, Vortical Flows and Vortex-Induced Vibrations*
Technical University of Denmark, Copenhagen, Denmark
vortex.compute.dtu.dk
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Vorticity Generation 2: Free surfaces

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Reading Material

Brøns, M., Thompson, M.C., Leweke, T. & Hourigan, K., Vorticity generation and conservation for two-dimensional interfaces and boundaries, Journal of Fluid Mechanics, 758, 63-93, 2014.

Lugt, H. J., Local flow properties at a viscous free surface, Phys. Fluids, 30, 3647-3652, 1987.

Lundgren, T. & Koumoutsakos, P., On the generation of vorticity at a free surface, J. Fluid Mech., 382, pp. 351-366, 1999.

Morton, B. R., The generation and decay of vorticity, Geophysical & Astrophysical Fluid Dynamics, 28:3, 277 - 308, 1984.

Reichl, P., Hourigan, K. & Thompson, M. C., Flow past a cylinder close to a free surface, J. Fluid Mech., 533, 269-296, 2005.

Rood, E. P., Myths, math, and physics of free-surface vorticity in "Mechanics USA 1994" edited by AS Kobayashi, Appl Mech Rev, 47(6), part 2, June 1994.

Sheridan, J., Lin, J.-C., & Rockwell, D., Flow past a cylinder close to a free surface, J. Fluid Mech., 330, 1-30, 1997.

Sarpkaya, T., Vorticity, free surface, and surfactants, Annu. Rev. Fluid Mech., 28, 83-128.

Wu, J. Z. & Wu, J. M., Vorticity dynamics on boundaries, Adv. Appl. Mech., 32, 119-275, 1996.

Lecture Objectives

Learn:

How does a stress-free boundary differ from a solid boundary

The solid-body rotation condition at a stress-free surface

How such boundaries can act as sources and sinks

That vorticity is still conserved with a stress-free surface

How does vorticity generation at a stress-free (“free”) surface differ from that at a solid surface

Flow velocity at surface is no longer “constant” (i.e., can slip)

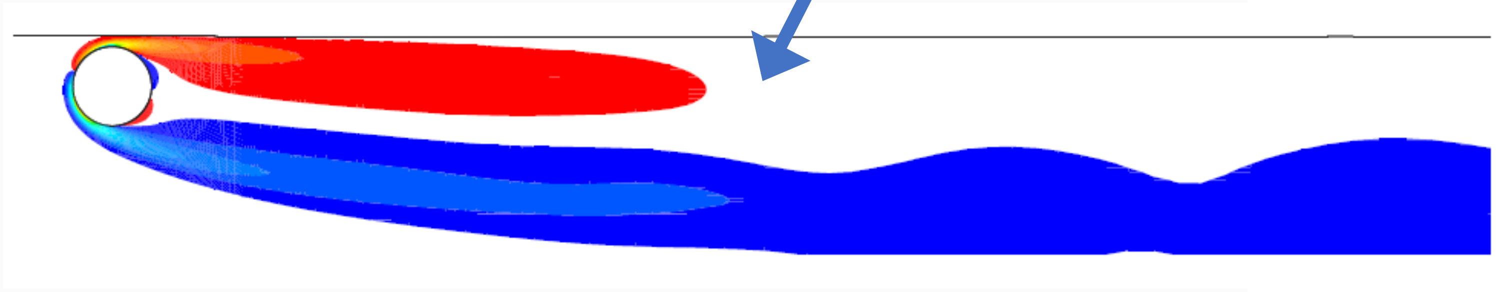
Surface can distort in shape (c.f. a solid interface)

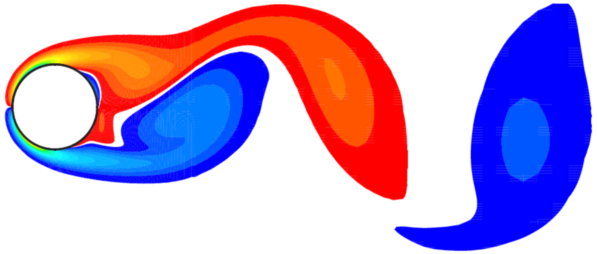
Froude number

$$Fr = U/\sqrt{(gL)}$$

U = flow velocity
L = characteristic dimension
g = gravitational constant

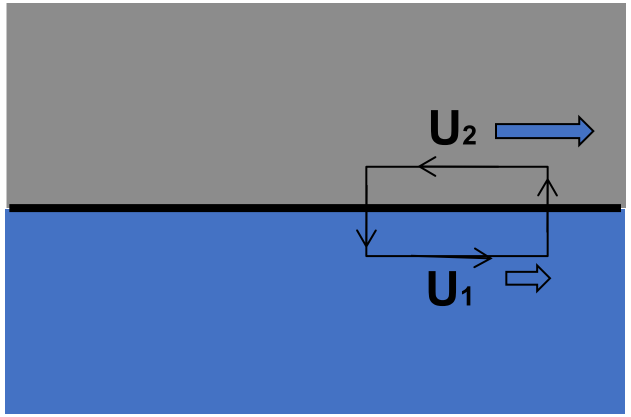
THE CURIOUS CASE OF THE VANISHING VORTICITY





Limiting case of an inviscid Fluid 2

U_2 , ρ_2, ν_2



Fluid 2

Interface

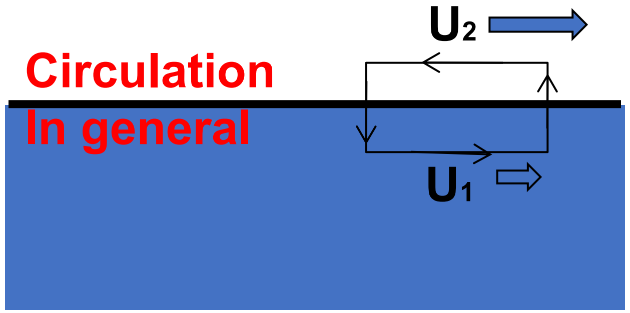
Fluid 1

U_1 , ρ_1, ν_1

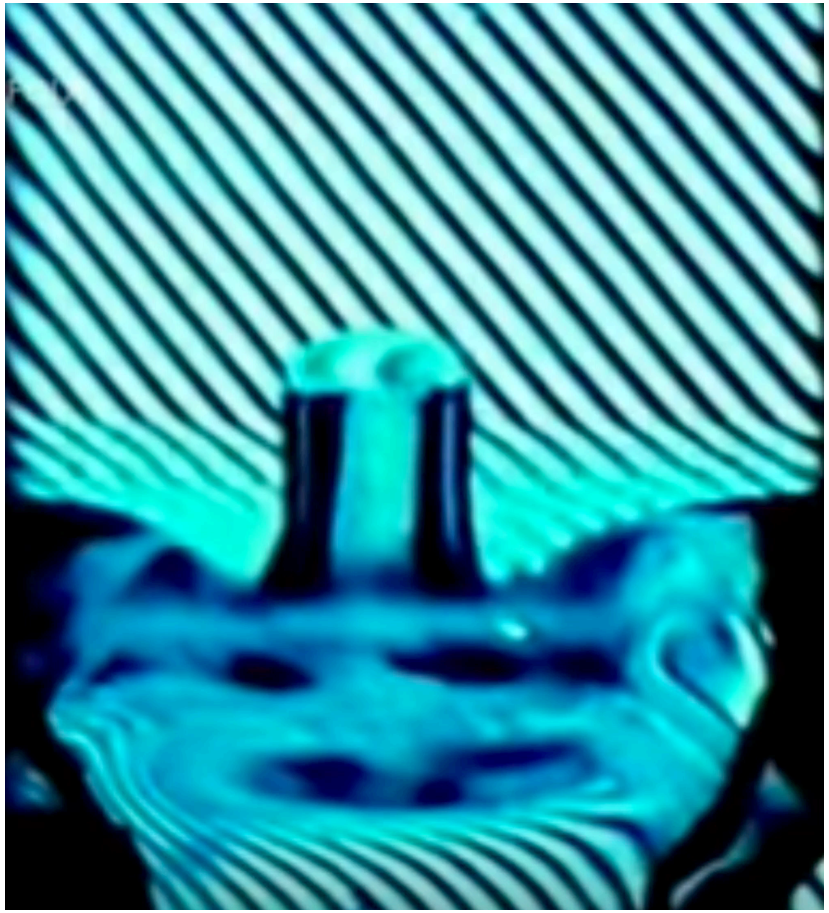
$$\Gamma = \oint_C \mathbf{V} \cdot d\mathbf{l}$$

Circulation at interface:
$\gamma = U_1 - U_2$

ν_2 -> *0*,
ρ_2 -> 0
Free surface



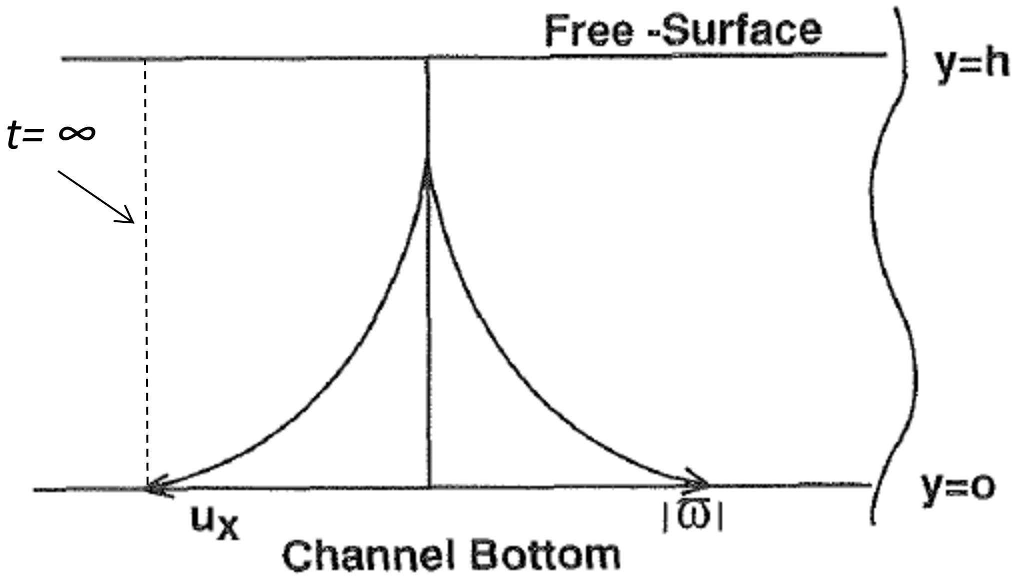
Superfluids have zero viscosity



Superfluid fountain – frictionless flow forever

www.youtube.com/watch?v=2Z6UJbwxBZI

A previous view of free surface channel flow



$$u_x = -U_0 + \sum_{k=0}^{\infty}\left\{C_k\, e^{-\left[\frac{\pi^2}{h^2}\left(\frac{2k-1}{2}\right)\right]^2 \nu t}\sin\left[\frac{\pi}{h}\left(\frac{2k-1}{2}\right)y\right]\right\}$$

"At *t*= ∞ , u_x = U_0 *everywhere in the* fluid. Momentum is conserved as the impulse of the plate is entirely transferred to the momentum of the water. However, the initial vorticity produced by the accelerated plate has entirely disappeared (the flow has constant velocity). "

"Notice that there is no requirement that a second fluid be present above the channel fluid to generate or absorb vorticity."

Rood, E. P. (1994) '*Myths, math and physics of free-surface vorticity*'

Vorticity is not conserved?

"Vorticity is neither conserved nor is it angular momentum. Therefore it is physically acceptable that vorticity entirely disappear from a flow. The disappearance of the vorticity is unsettling if vorticity is erroneously ascribed the same conservation principle that applies to momentum."

Rood (1994)

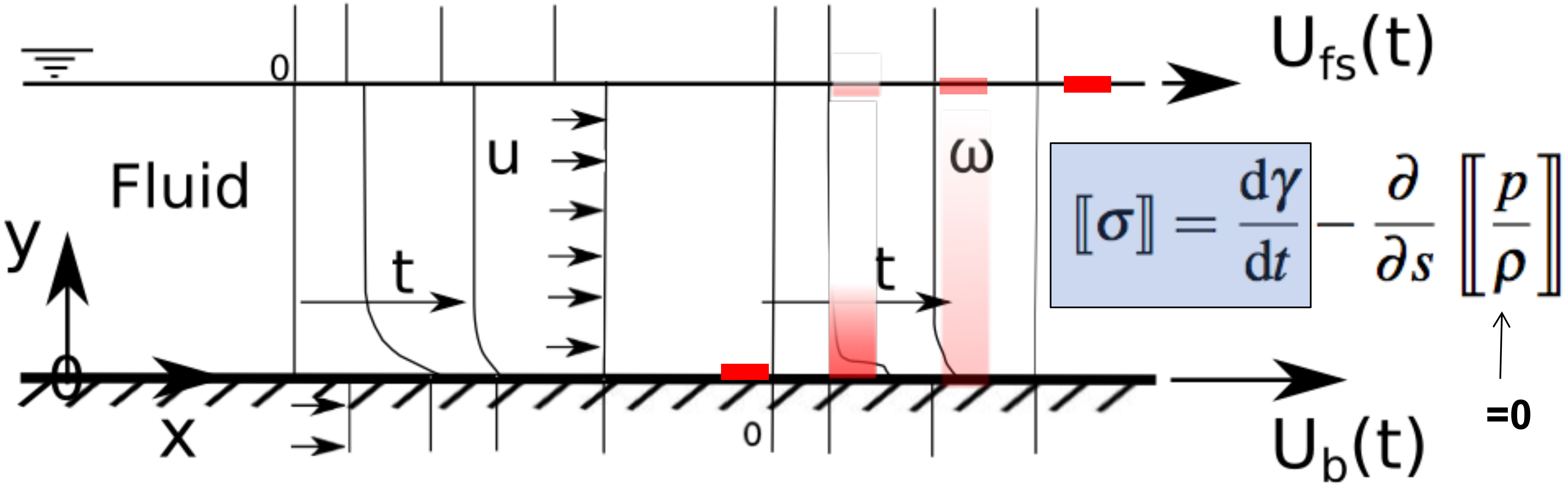
Vorticity is not conserved?

Vorticity can disappear from a fluid through a free surface?

From a physicist's point of view, conservation principles are very important.

Here, we will look for conservation of vorticity.

Plane Couette Flow:
In the limit, all vorticity will reside at a stress-free surface above



At t = 0, all the vorticity is generated by acceleration at solid wall.

t > 0, vorticity diffuses away from the boundary and approaches zero in the limit in the body of the flow.

Vorticity is conserved by the velocity jump/vortex sheet at the free surface.

MORTON: Generation of vorticity at a solid boundary is an inviscid process

$$\sigma = \frac{d\boldsymbol{u}}{dt} \cdot \hat{\boldsymbol{t}} + \frac{\partial}{\partial x}\frac{p}{\rho} - \boldsymbol{g} \cdot \hat{\boldsymbol{t}}$$

Rate of vorticity generation = (Acceleration of lower boundary) – (Inviscid acceleration of fluid elements due to pressure/body forces)

Morton, B.R (1984) *'The generation and decay of vorticity'*

$$\delta k = \oint_{\mathscr{C}} \mathbf{v} \cdot d\mathbf{r} \approx (u - U)\delta x$$

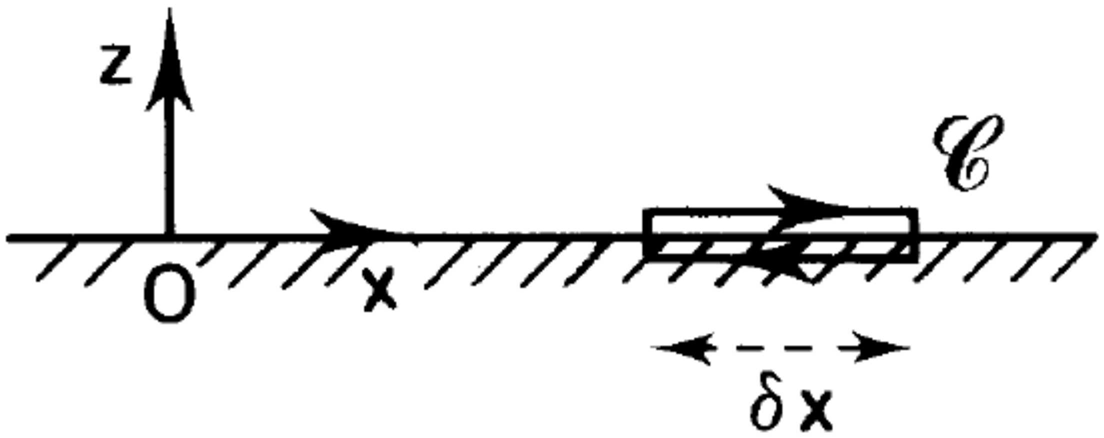
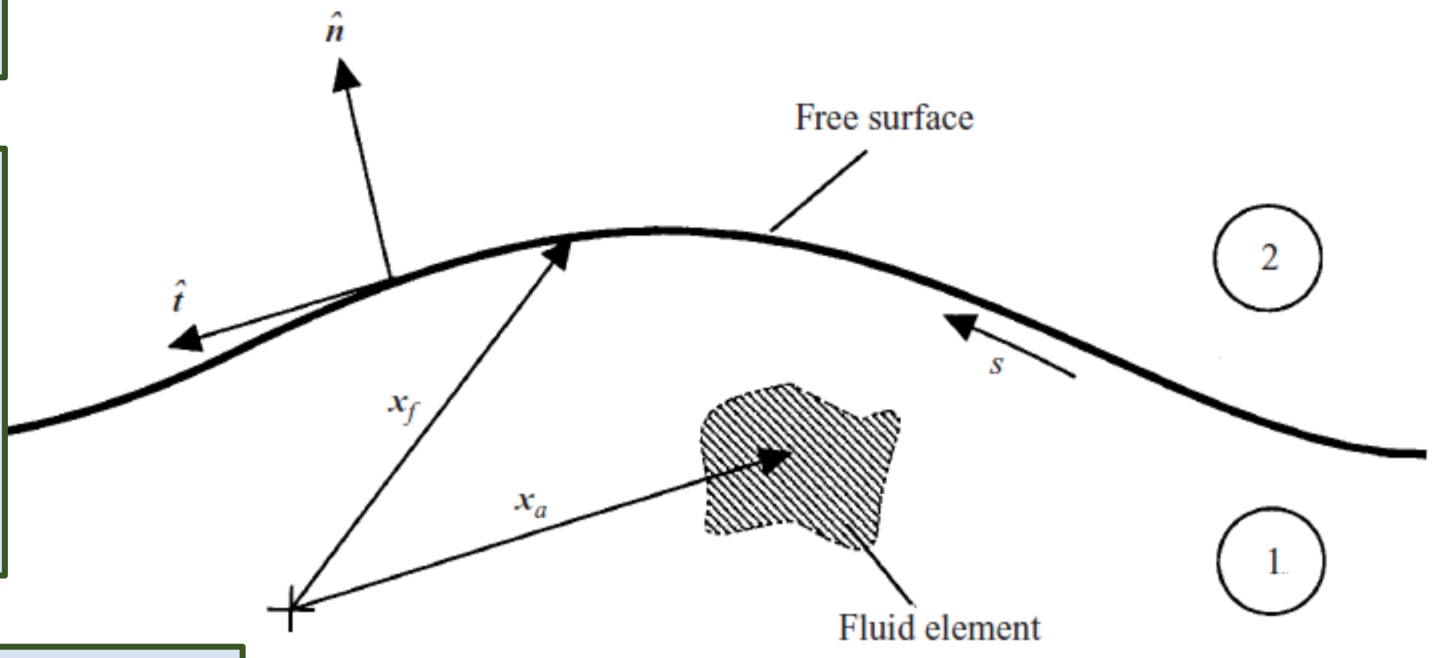


FIGURE 9 The circulation circuit $\mathscr{C}$ for generation at a boundary.

The vorticity flux at a free-surface is due to viscous effects

$$\sigma = \frac{d\boldsymbol{u}}{dt} \cdot \hat{\boldsymbol{t}} + \frac{\partial}{\partial s}\frac{p}{\rho} - \boldsymbol{g} \cdot \hat{\boldsymbol{t}}$$

Rate of vorticity generation = (**Total acceleration of Fluid**) – (**Inviscid acceleration of fluid elements due to pressure/body forces**)
= **Tangential Viscous acceleration of boundary fluid**



**LUNDGREN & KOUMOUTSAKOS,
On the generation of vorticity at a free surface,
Journal of Fluid Mechanics, Volume 382, March 1999, pp 351-366**

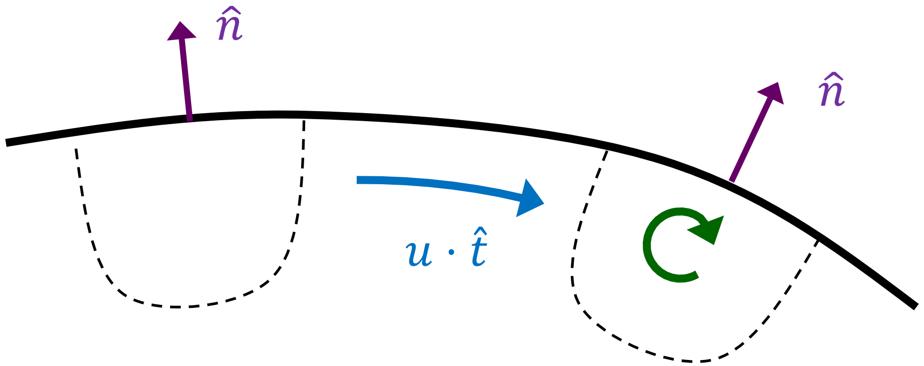
The shear-free condition constrains boundary vorticity at the free-surface

2D: $$\omega_s = 2\kappa(\boldsymbol{u}\cdot\hat{\boldsymbol{t}}) - 2\frac{\partial}{\partial s}(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})$$

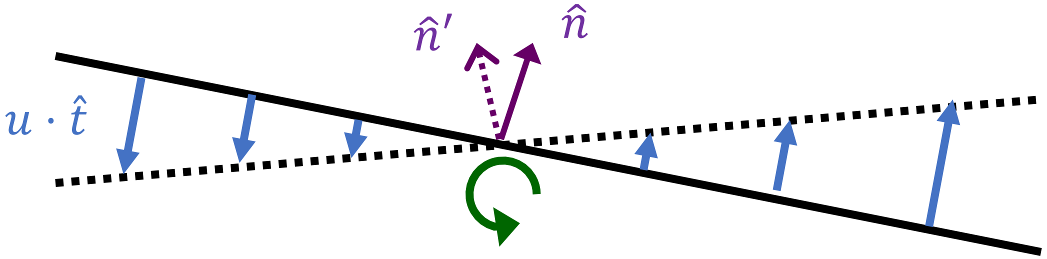
3D: $$\boldsymbol{\omega}_{s,\pi} = -2\times(\boldsymbol{u}\cdot\boldsymbol{K} + \nabla_\pi \boldsymbol{u})$$

Fluid elements on the interface rotate at the same rate as the normal vector on the interface

$$\omega_s = 2\frac{d\hat{\boldsymbol{n}}}{dt}\cdot\hat{\boldsymbol{t}}$$

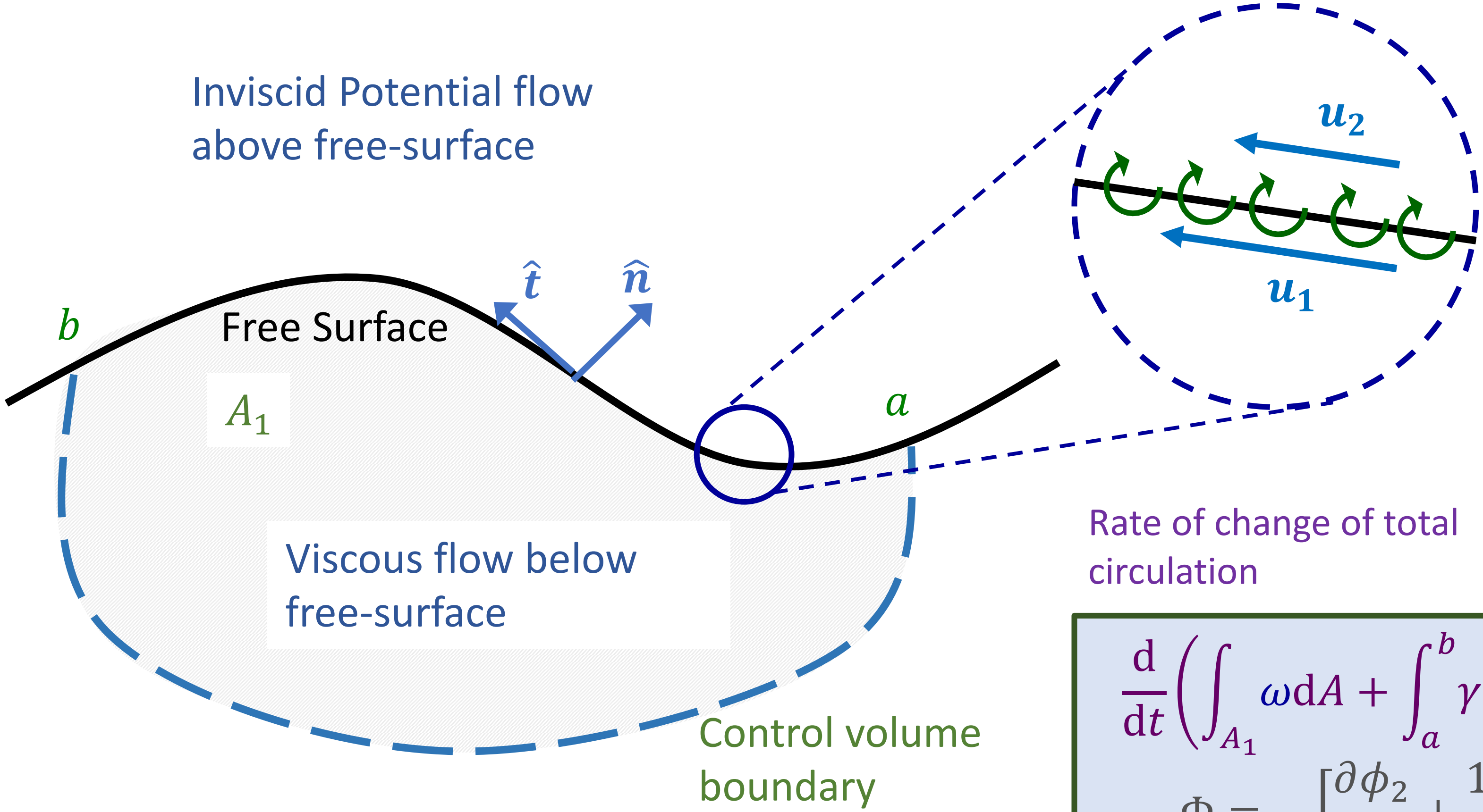


'Solid Body' rotation of fluid along interface



Rotation of fluid element due to interface normal motion

Vorticity is conserved at a free-surface



$\gamma = (\boldsymbol{u_2} - \boldsymbol{u_1}) \cdot \hat{\boldsymbol{t}}$: Circulation contained in the interface vortex sheet

Rate of change of total circulation

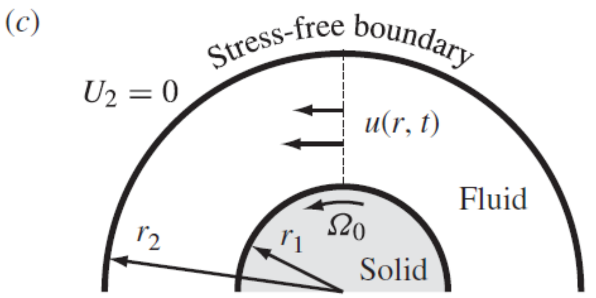
Circulation transport along interface

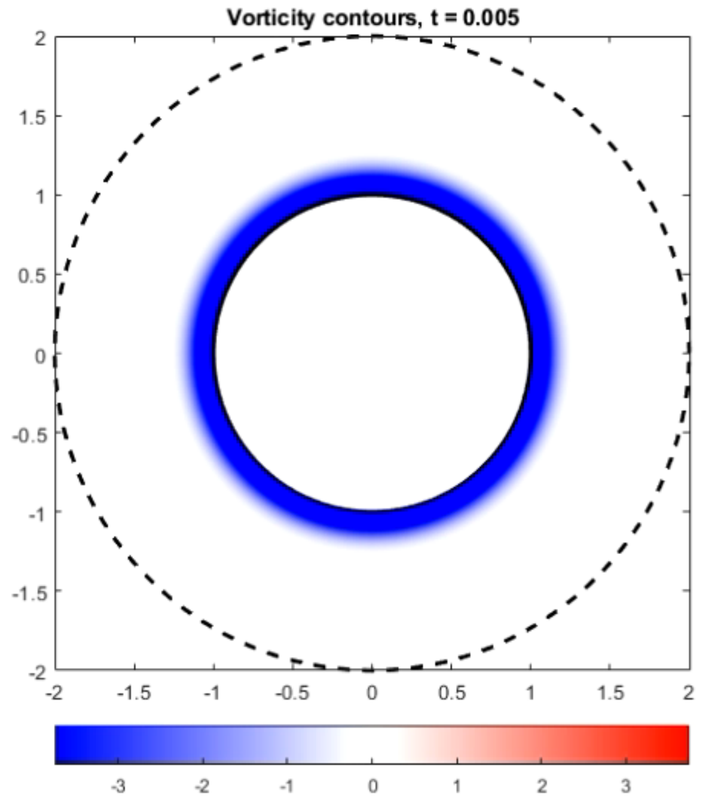
Vorticity diffusion out of control volume

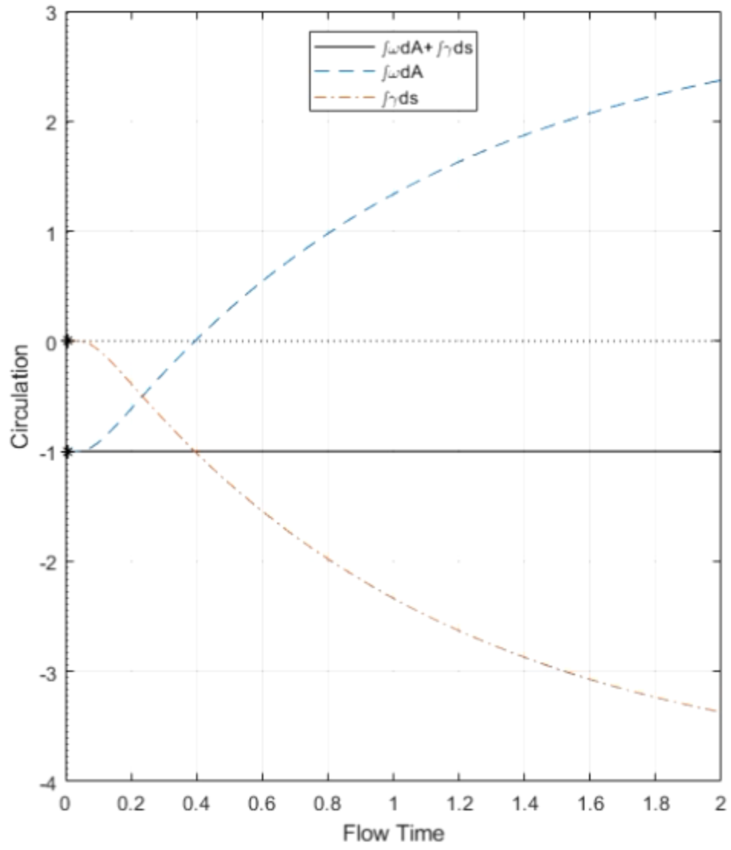
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{A_1} \omega \mathrm{d}A + \int_a^b \gamma \mathrm{d}s\right) = \Phi_a - \Phi_b + \int_{C_1} \nu \frac{\partial \omega}{\partial n} \mathrm{d}s$$

$$\Phi = -\left[\frac{\partial \phi_2}{\partial t} + \frac{1}{2}\boldsymbol{u_2} \cdot \boldsymbol{u_2} - \frac{1}{2}\gamma^2 + gy\right] - \frac{p_1}{\rho}$$

EXAMPLE: Axisymmetric flow beneath a free-surface



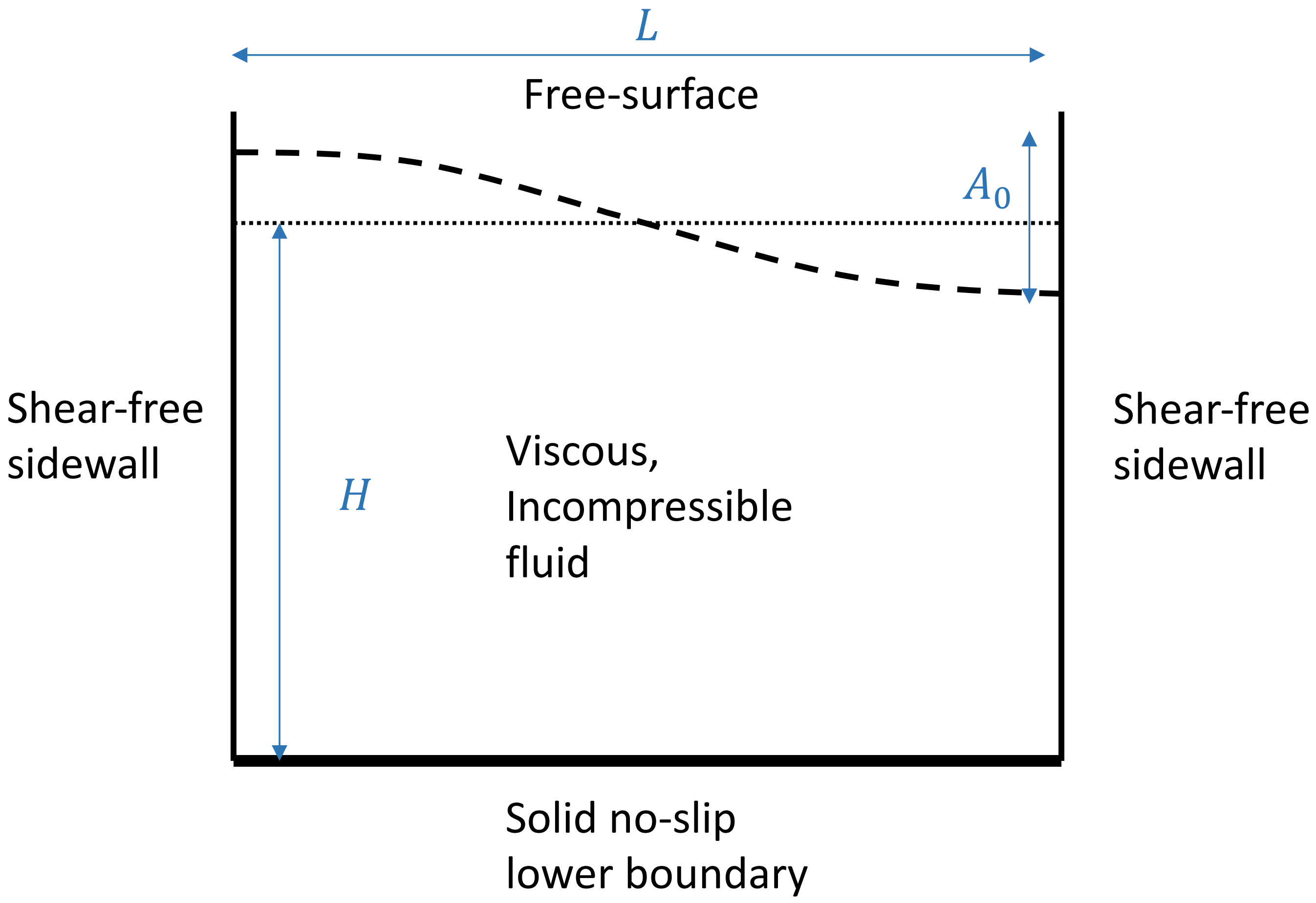




Brøns, M., Thompson, M. C., Leweke, T., and Hourigan, K. (2014) *'Vorticity generation and conservation for two-dimensional interfaces and boundaries'*

Vorticity Generation:
Sloshing motion at a free-surface

Problem Description



PARAMETERS USED:

$L = 2$

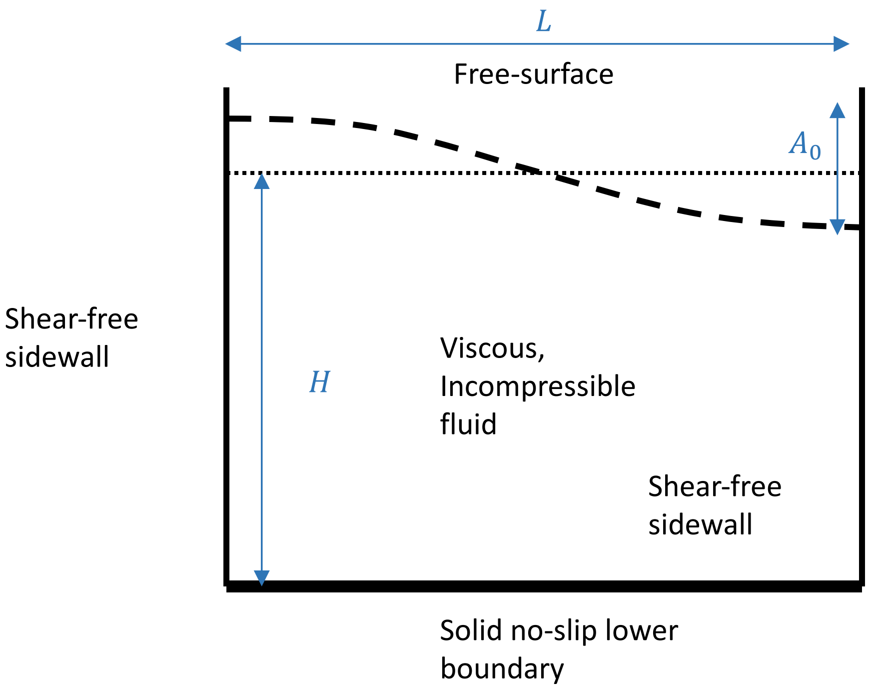
$A_0 = 0.2$

$H = 10$

$\nu = 0.01$

$\rho = 1$

Circulation Balance



Rate of change of total circulation: includes contribution from fluid interior and free surface vortex sheet

$$\frac{d}{dt}\left(\int_A \omega \mathrm{d}A - \int_a^b u \cdot \hat{t}\,\mathrm{d}s\right) =$$

Diffusion of circulation through outer boundary: Generation of vorticity at lower boundary, and diffusion of vorticity into the shear-free solid boundaries

$$\int_{C_1} \nu \hat{\boldsymbol{n}} \cdot \nabla\omega\,\mathrm{d}s + \int_{C_1} \hat{\boldsymbol{n}} \cdot (\boldsymbol{v}_b - \boldsymbol{u})\omega \mathrm{d}s$$

Advection of vorticity through outer boundary: Is zero as $\boldsymbol{u} \cdot \hat{\boldsymbol{n}} = \boldsymbol{0}$ lower and side boundaries

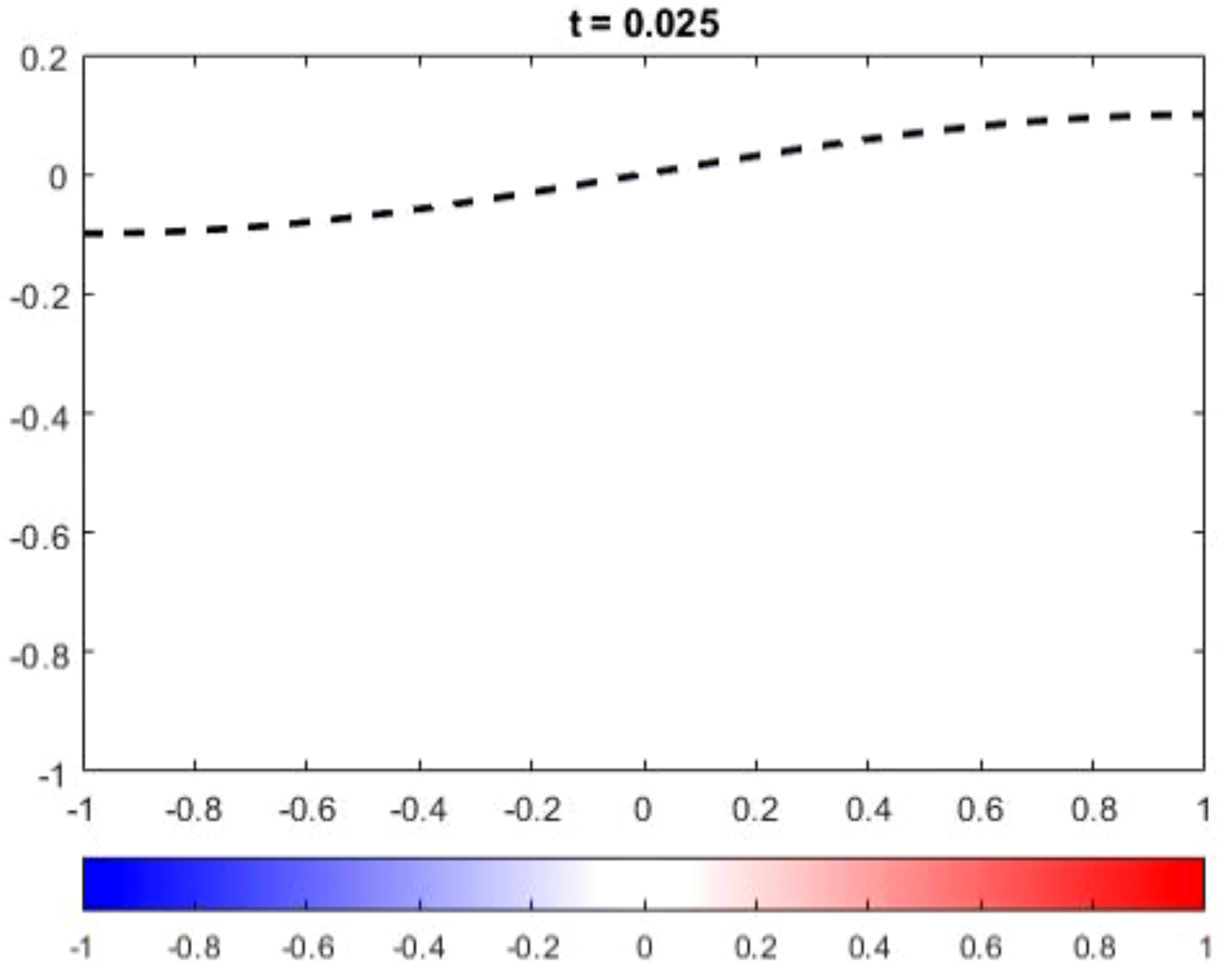
Generation of interface circulation: By pressure gradients and body forces

$$+\left.\frac{p}{\rho}\right|_b - \left.\frac{p}{\rho}\right|_a + [\mathrm{gz}]_\mathrm{b} - [\mathrm{gz}]_\mathrm{a}$$

Velocity Source Terms: Transport of interface circulation into the control volume along the free-surface

$$+\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\Big|_b - \frac{1}{2}\,\boldsymbol{u}\cdot\boldsymbol{u}\Big|_a - \boldsymbol{u}\cdot\boldsymbol{v}_b\Big|_b + \boldsymbol{u}\cdot\boldsymbol{v}_b\Big|_a$$

Solution Animation

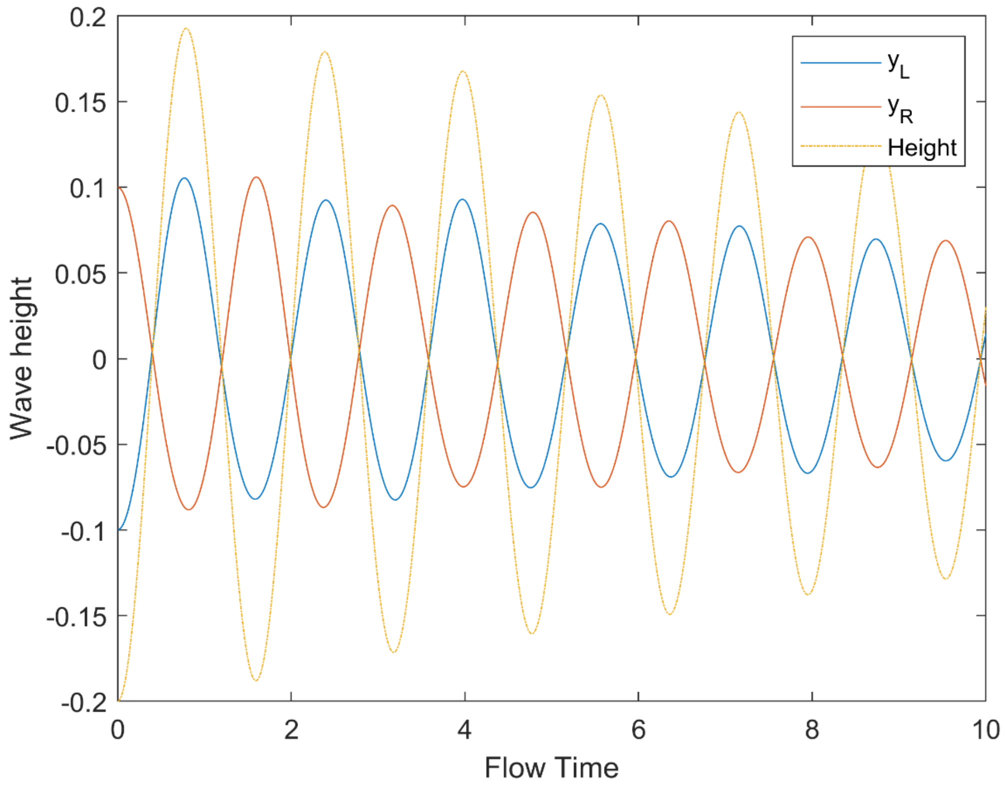


Vorticity appears spontaneously on the free-surface to ensure the rotation rate of fluid elements on the surface is equivalent to solid body rotation.

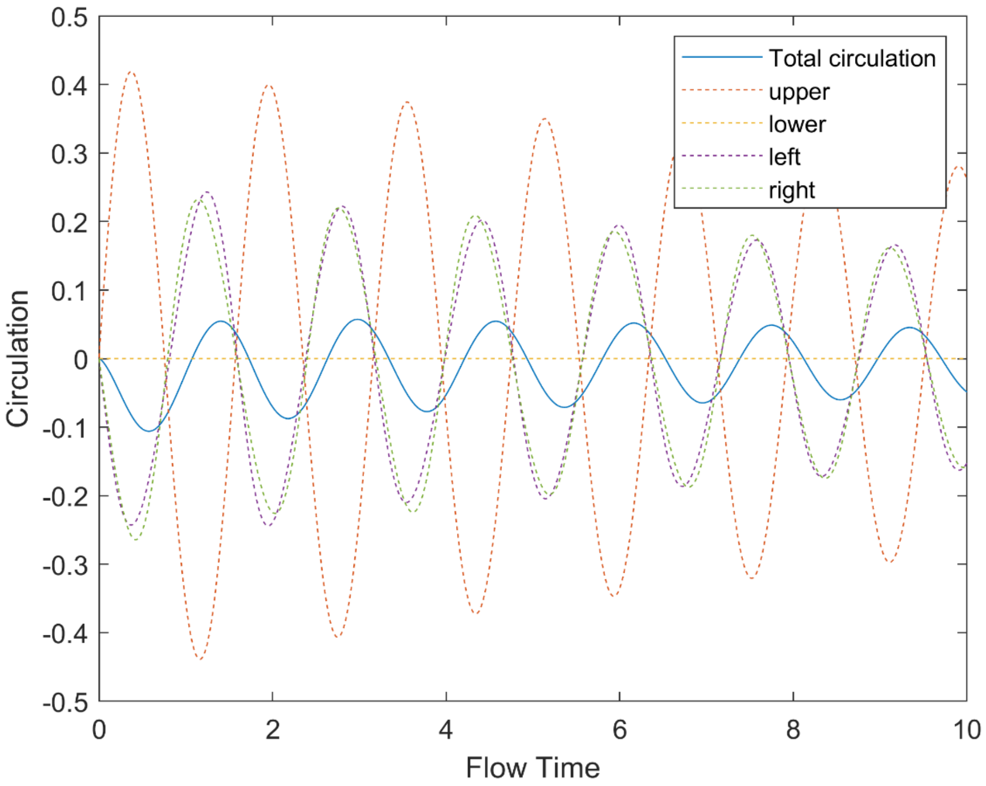
Vorticity diffuses into the fluid under the action of viscous forces

Oscillatory motion of the free surface produces alternate shedding of positive and negative signed vorticity into the flow.

Global Circulation Balance



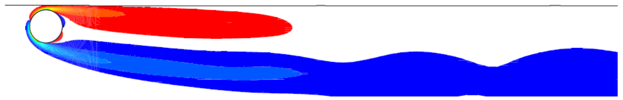
Decay of wave amplitude with time



Contributions of interface and sidewalls to total circulation ($\int \omega dA$)

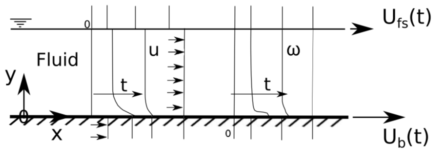
Outline of Vanishing Vorticity Problem

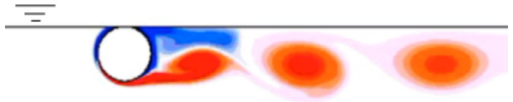
- The Dilemma



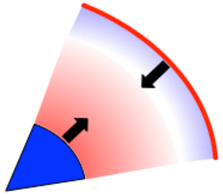
$$[\![\sigma]\!] = \frac{d\gamma}{dt} - \frac{\partial}{\partial s}\left[\!\left[\frac{p}{\rho}\right]\!\right]$$

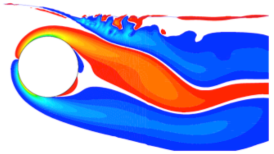
- Flat Surface





- Curved Surface

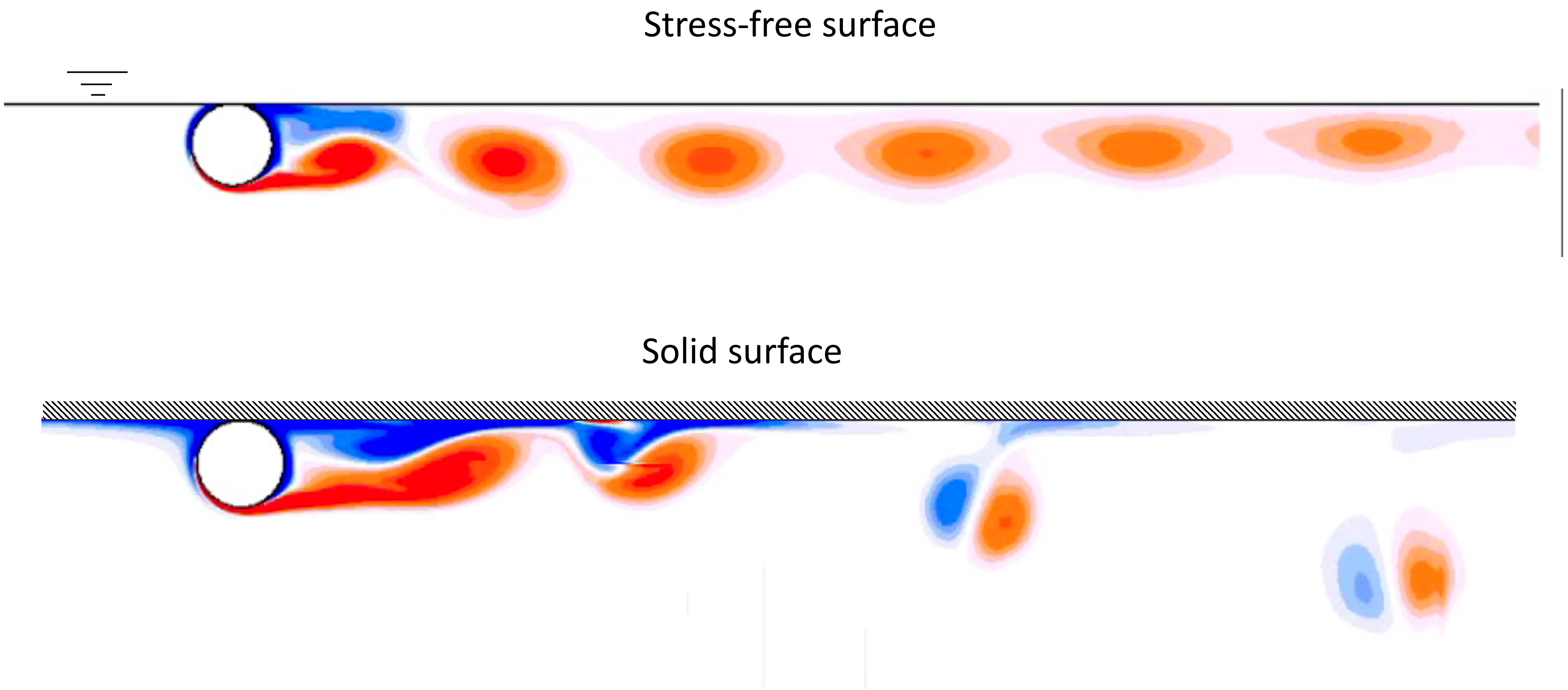




- The Resolution

Free surface circulation + cross-annihilation

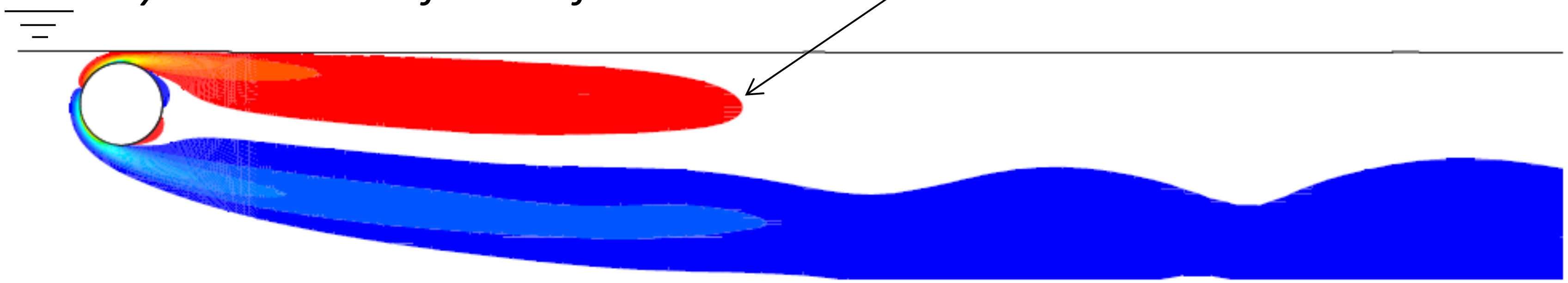
The Dilemma: Vorticity of one sign vanishes near a stress-free surface (but not a solid surface)



First, what happens for a flat stress-free surface?

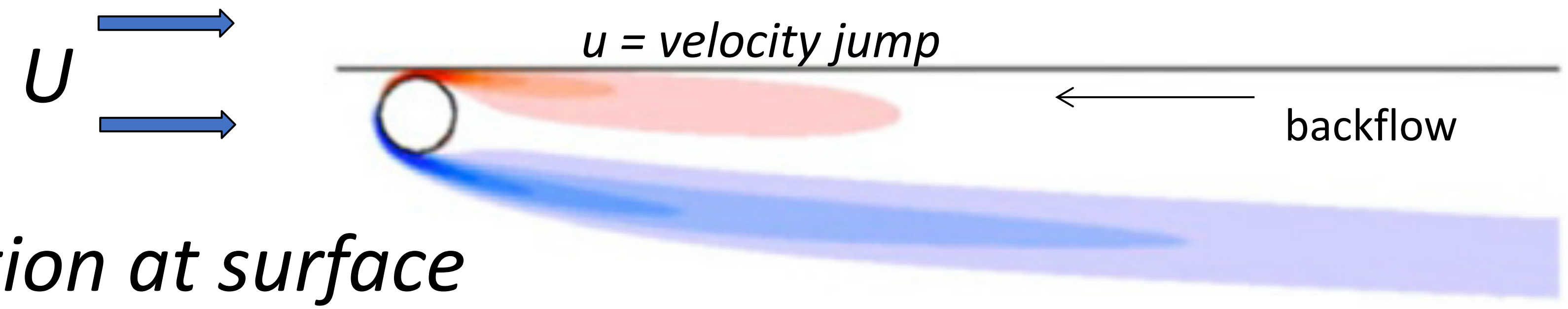
Cylinder near a free surface

Why does this vorticity disappear?

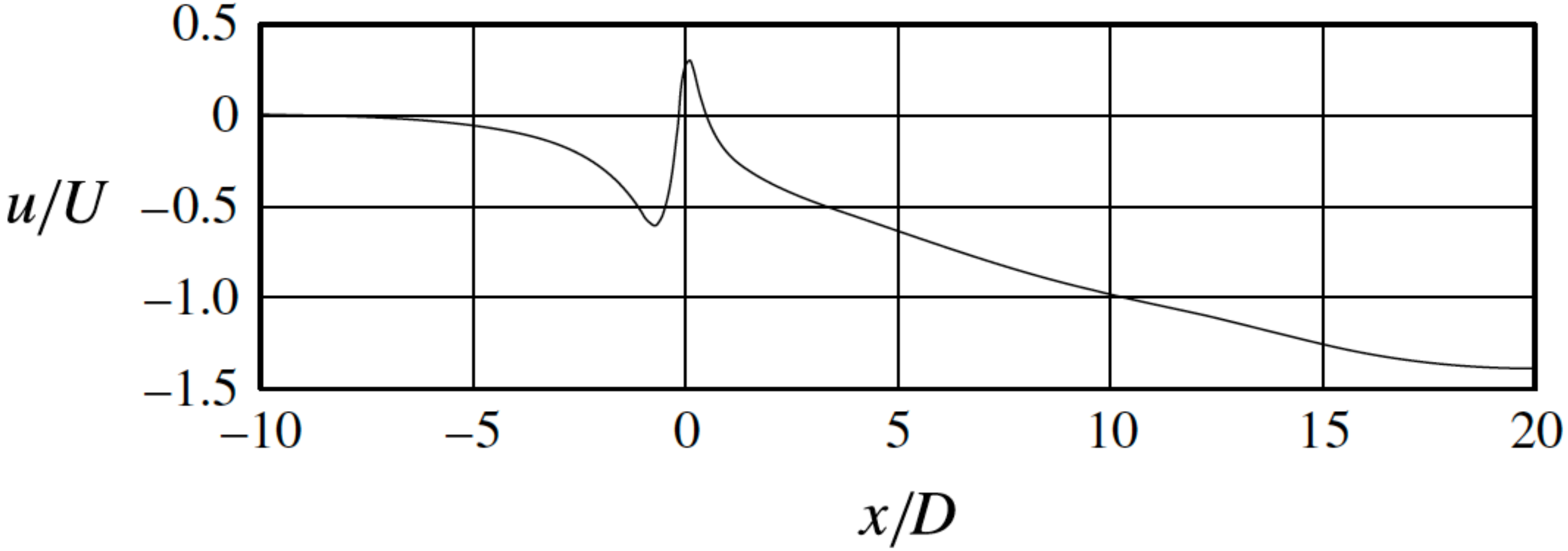


Reichl, P., Hourigan K. & Thompson, M.C., JFM, 533, 269 – 296, 2005.

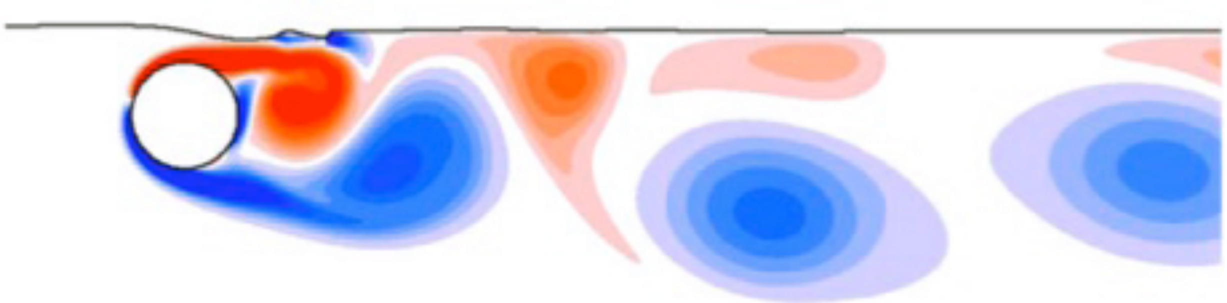
“Loss” of vorticity in body of flow is matched by “addition” of circulation at stress-free surface



Circulation at surface



But what happens if there is surface curvature?



At a (steady) stress-free boundary in a fluid:
flow must locally be in solid body rotation

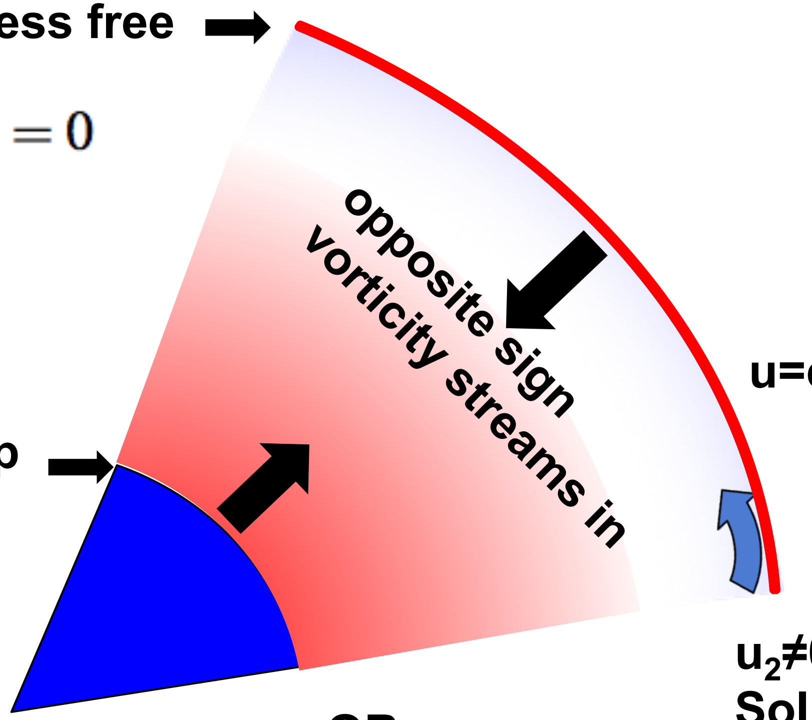
$$\omega_s = 2u/R_c$$

Consider simple case Spinning cylinder: outer stress-free surface

stress free

$$\frac{\partial u}{\partial n} = 0$$

no slip



$u_1 = \Omega R_1$

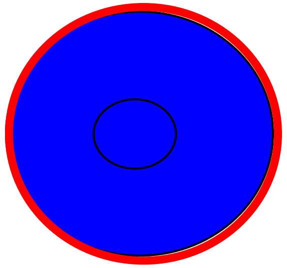
u=const

$$\boxed{[\![\sigma]\!] = \frac{d\gamma}{dt}} - \frac{\partial}{\partial s}\left[\!\!\left[\frac{p}{\rho}\right]\!\!\right]$$

$$\gamma = -[\![\boldsymbol{u} \cdot \boldsymbol{t}]\!]$$

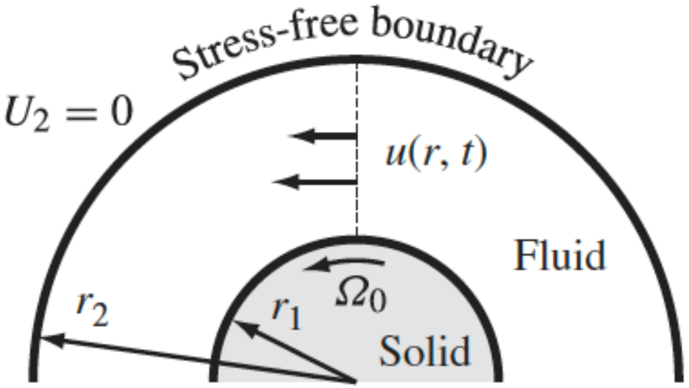
$u_2 \neq 0$, $\omega = 2u_2/R_2$
Solid body
rotation

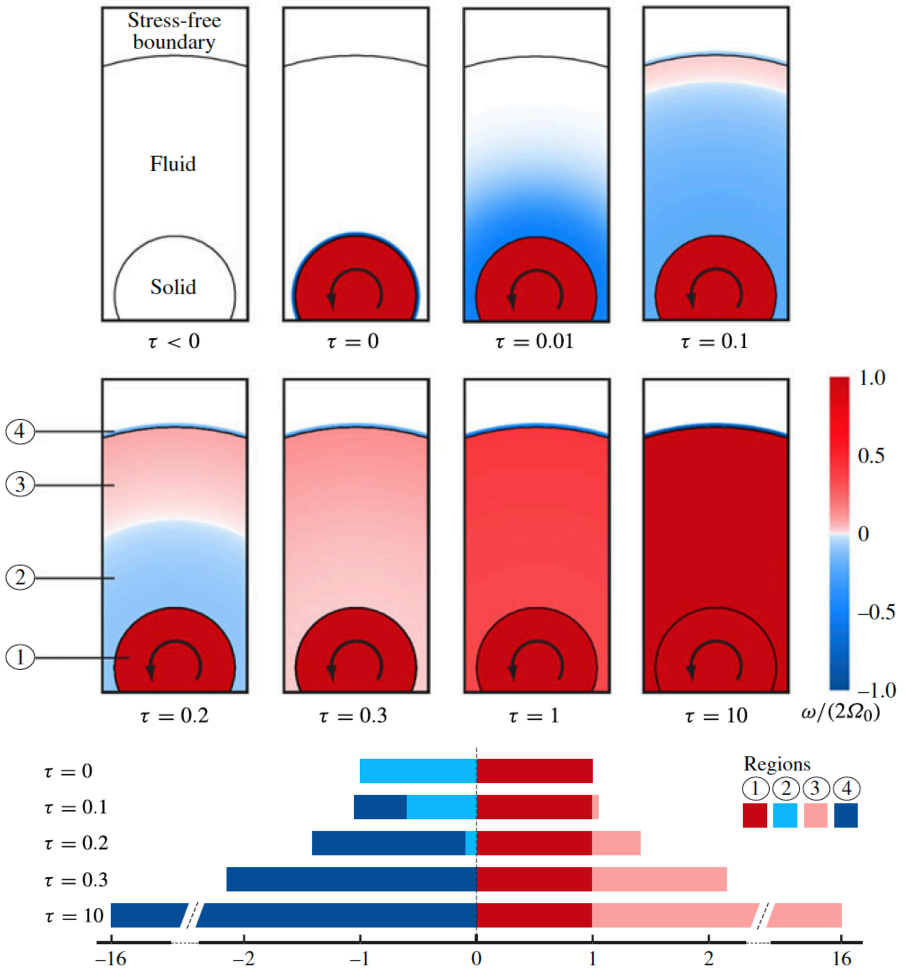
T → ∞



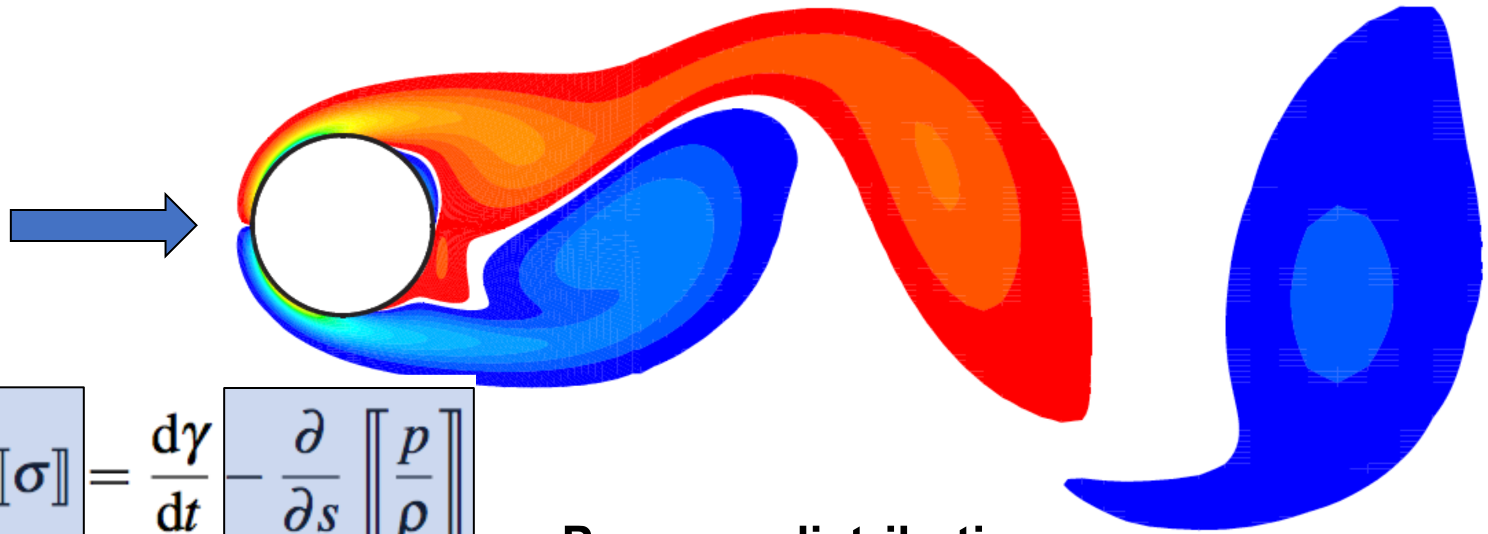
Solid body rotation
of cylinder + fluid
Vortex ring of opposite
sign at free surface

Strong flux of opposite sign vorticity from outer stress-free surface



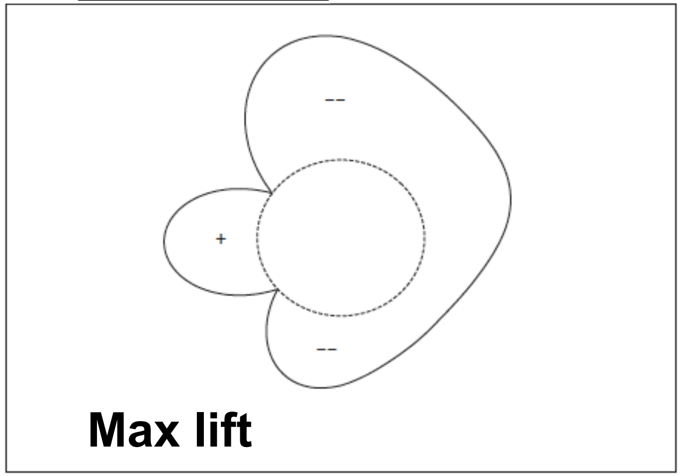


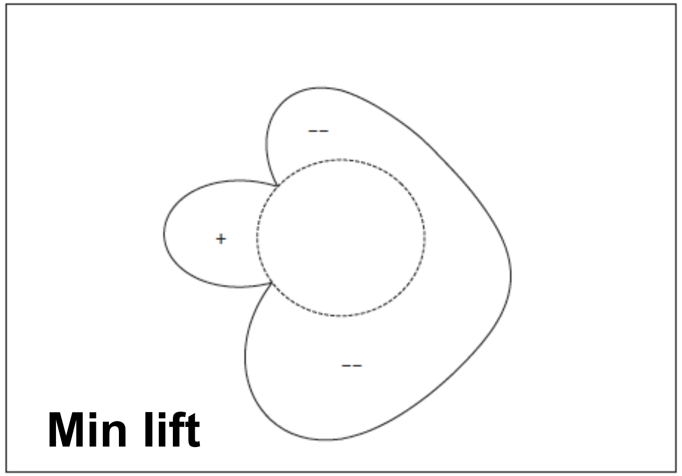
Vorticity generation from cylinder in flow



$$\boxed{[\![\sigma]\!]} = \frac{\mathrm{d}\gamma}{\mathrm{d}t} \boxed{- \frac{\partial}{\partial s}\left[\!\!\left[\frac{p}{\rho}\right]\!\!\right]}$$

Pressure distribution

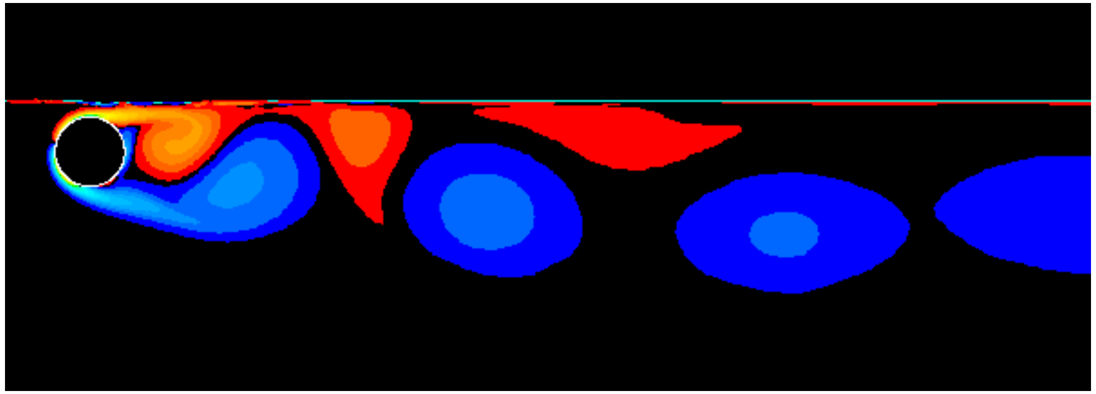




For non-axisymmetric case:

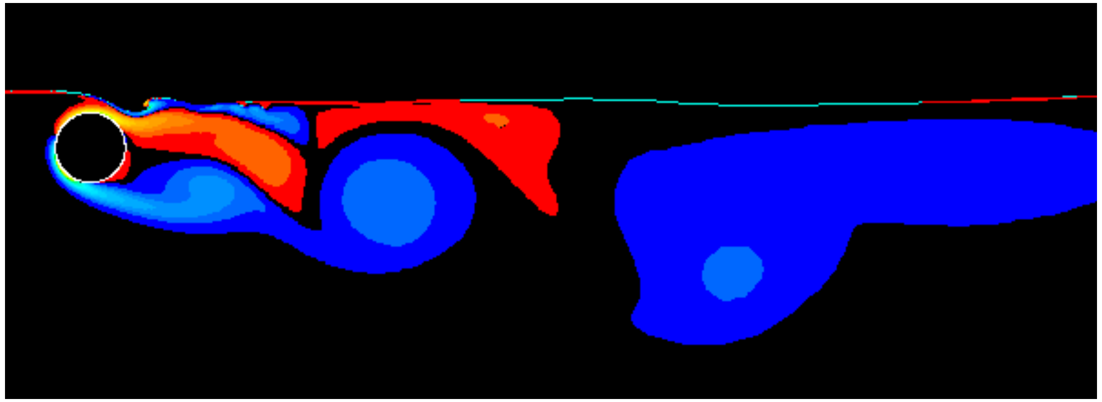
Curvature of stress-free surface leads to augmented cross-annihilation of vorticity

Surface curvature can lead to more rapid vorticity "disappearance"



Fr = 0.2

G/D = 0.2, Re = 180

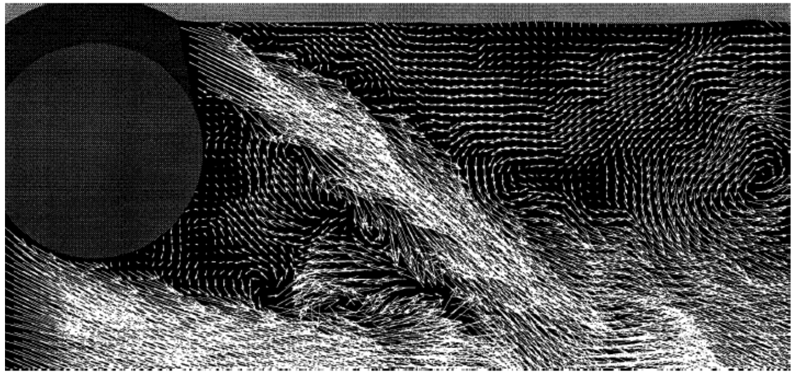


Fr = 0.5

G/D = 0.2, Re = 180

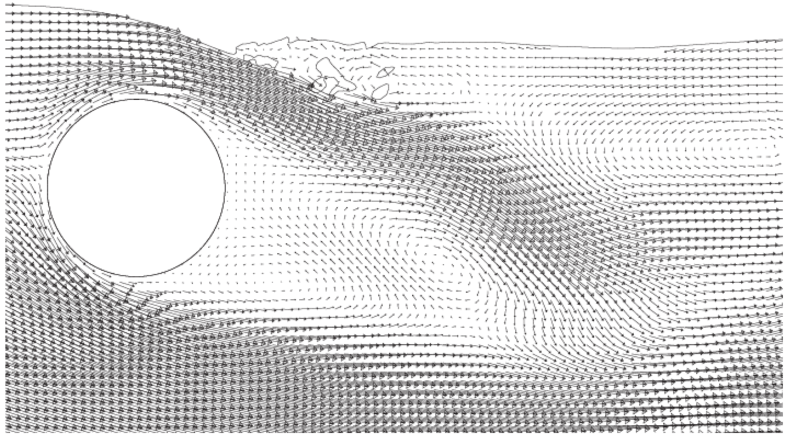
P. Reichl, PhD Thesis, Monash

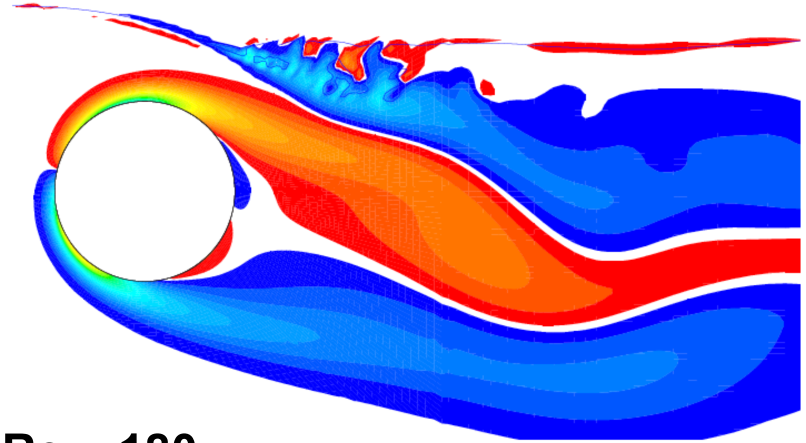
Larger surface curvature: stronger vorticity separation



Experiments: G/H = 0.43, Fr = 0.6, Re ~ 6-9x10^3

Sheridan et al., JFM, 330, 1–30, 1997.



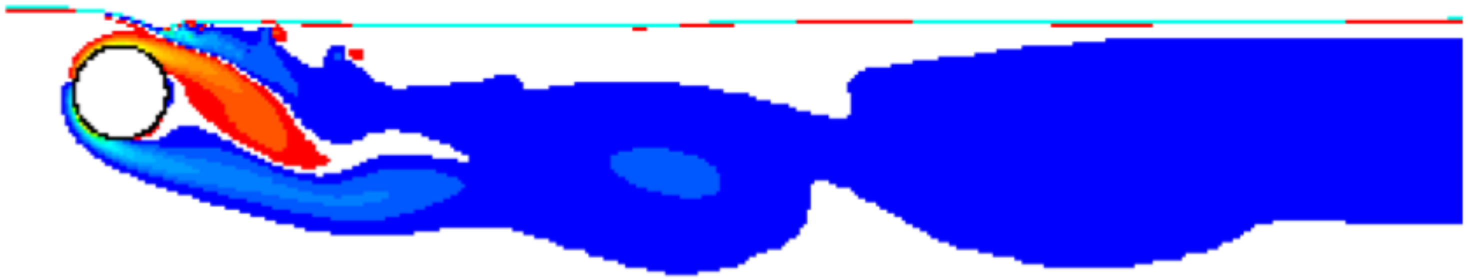


Predictions: G/H = 0.4, Fr = 0.6, Re ~ 180

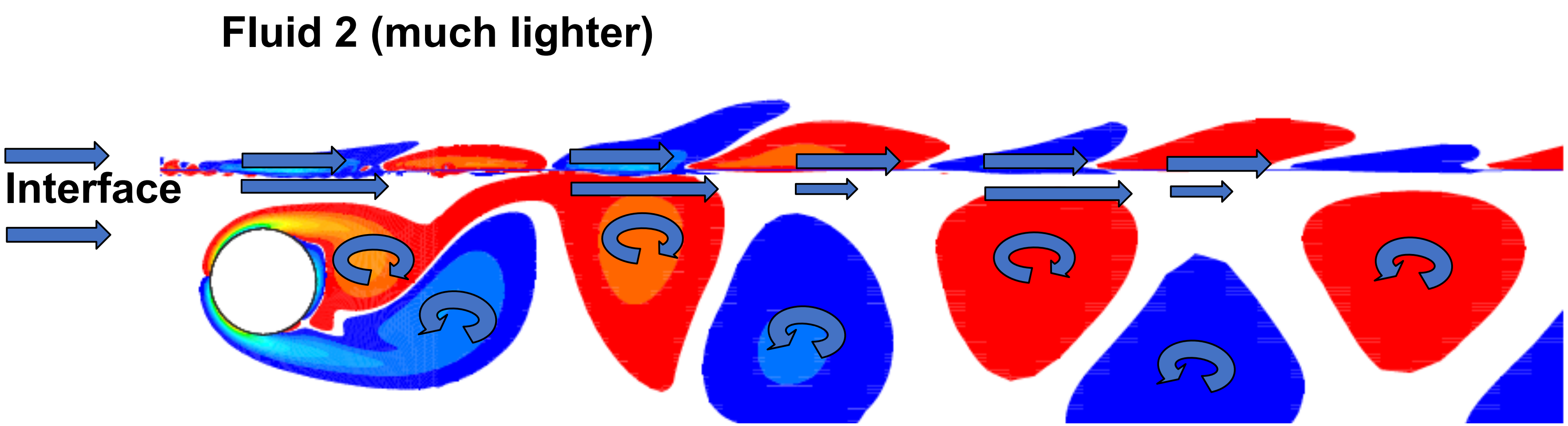
Reichl et al., JFM, 533, 269-296, 2005.

Vorticity shed from top of cylinder is cross-annihilated from both sides!

G/D = 0.1, Fr = 0.25, Re = 180



Two-fluid layers: vorticity in ‘vortex sheet’ diffuses out



$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_A \omega\,\mathrm{d}A + \int_a^b \gamma\,\mathrm{d}s\right)} = \oint_C \nu\nabla\omega\cdot\hat{\boldsymbol{n}}\,\mathrm{d}s - \left(\left[\!\left[\frac{p}{\rho}\right]\!\right]_b - \left[\!\left[\frac{p}{\rho}\right]\!\right]_a\right)$$

$$+ \int_a^b \kappa\left[\!\left[(\boldsymbol{u}\cdot\boldsymbol{n})(\boldsymbol{u}\cdot\boldsymbol{t})\right]\!\right]\,\mathrm{d}s - \frac{1}{2}\left(\left[\!\left[(\boldsymbol{u}\cdot\boldsymbol{n})^2\right]\!\right]_b - \left[\!\left[(\boldsymbol{u}\cdot\boldsymbol{n})^2\right]\!\right]_a\right)$$

P. Reichl, PhD Thesis, Monash

Conclusions

- Vorticity is the essence of bluff body wakes
- To control drag, lift, vibrations, etc, it helps to understand where and how vorticity is generated
- Vorticity is a physical quantity that is conserved (and for some important cases, produced all in first instant, cf angular momentum)
- Stress-free surfaces can be sinks and sources of vorticity
- For higher Froude numbers (larger surface distortions), separation of flow at free surface leads to further cross-annihilation