Lecture 4

Long- and short-wave instabilities

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Long-wave instabilities of a vortex / system
- displacement / deformation of the vortex shape
- description by a filament approach

Short-wave instabilities
- internal deformations of the vortex core
- interaction of vortex Kelvin modes

Vortex pairs

Helical vortices
Long-wave instability
Vortex pairs

Crow instability
Visualisations of aircraft trailing wakes
SEMPRE A BORDO. SEMPRE REFRESCANTE.
Long-wavelength Crow instability
Crow instability

Only natural way for aircraft wake decay in a calm atmosphere
Dynamics of two point vortices
- same circulation -

counter-rotating

co-rotating

mutually induced velocity:
\[ |V| = \frac{\Gamma}{2\pi b} \]
Vortex pair parameters

**counter-rotating**

- vortex separation $b$
- descent speed $V = \Gamma / 2\pi b$
- non-dim. time $t^* = t V / b$

**co-rotating**

- vortex separation $b$
- angular freq. $\Omega = (\Gamma_1 + \Gamma_2) / 2\pi b^2$
- non-dim. time $t^* = t \Omega / 2\pi$

rate of strain induced by one vortex on the other: $\varepsilon = \Gamma / 2\pi b^2$
Counter-rotating vortex pair

Basic state:
vertical translation with speed \( V = \frac{\Gamma}{2\pi b} \)
**Counter-rotating vortex pairs**

**Experimental set-up**

- water tank (180 × 45 × 60 cm³)
- starting vortices generated by impulsively rotated plates
- computer-controlled plate motion

**Methods:**
- visualisation
- image analysis
- Particle Image Velocimetry (PIV)
Crow instability \((Re = 1500–2500, a/b \approx 0.2)\)
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Crow instability ($Re = 1500–2500, a/b \approx 0.2$)
**Crow instability mechanism**

Consider:

- two vortex filaments (core size $a$, separation $b$)
- plane sinusoidal displacement perturbations (wavelength $\lambda$, orientation $\theta$)

Symmetric modes

Anti-symmetric modes

\[
\hat{r}_1 = \hat{r}_1 \sin \left( \frac{2\pi x}{\lambda} \right) \exp(\alpha t)
\]
Crow instability mechanism

Evolution of perturbation
(example: right vortex):

self-induced motion

rotation of the plane
\[ f(a, \lambda) \]

strain from left vortex

rotation + stretching
\[ f(\theta) \]

mutual induction

rotation + stretching
\[ f(\theta, \lambda) \]

IF

- total rotation = 0
- total stretching > 0

Instability
Crow instability mechanism

Find $\alpha$:

- Biot-Savart line integrals
- cut-off method for self-induction
- linearisation

$$(\alpha^*)^2 = \left[ 1 - (kb_o)^2 K_0(kb_o) - kb K_1(kb_o) - \frac{\sigma}{(a_e/b_o)^2} \right] \left[ 1 + kb_o K_1(kb_o) + \frac{\sigma}{(a_e/b_o)^2} \right]$$

$\alpha^* = \alpha \left( \frac{2\pi b^2}{\Gamma} \right) \quad k = \frac{2\pi}{\lambda} \quad \sigma$: self-induced rotation rate
Self-induced rotation of a sinusoidal vortex filament
(Rankine vortex, Kelvin 1880)

\[ \sigma = \frac{1}{2} (ka)^2 \left[ \ln \frac{2}{ka} - C + \frac{1}{4} \right] \]
Stability diagram for a pair of counter-rotating vortices
(symmetric mode)
Stability diagram for a pair of counter-rotating vortices (symmetric mode)
Growth rate of the Crow instability

\( a/b = 0.0985 \)
Generalization to vortices other than Rankine

**Equivalent core size**
(Widnall et al. 1971)

\[
a_e = a \cdot \exp \left[ \frac{1}{4} - A + C \right]
\]

\[
A = \lim_{r \to \infty} \left[ \frac{4\pi^2}{\Gamma^2} \int_0^r \bar{r} v^2_{\phi}(\bar{r}) d\bar{r} - \ln \frac{r}{a} \right]
\]

\[
C = \frac{8\pi^2}{\Gamma^2} \int_0^\infty \bar{r} v_z^2(\bar{r}) d\bar{r}
\]

valid for \( \lambda \gtrsim 15 \ a \)
Crow instability *(measurements)*

- Uses $a/b_o$ and vortex velocity profile
- Growth rate predicted

(Crow 1970, Widnall et al. 1971)
Crow instability *(long-term evolution)*

Photographs of aircraft wake (Crow 1970)
Tip vortices of a Hawker Sea Fury
AIR14 air show at Payern Switzerland, 7 September 2014
(video by G. Balestra)
Long-wave instability
Vortex pairs

“Crow” instability
of 4-vortex systems
Co-rotating 4-vortex system
(Crouch, Boeing)
Counter-rotating 4-vortex system
(Jacquin et al., ONERA)

- outbound flaps
- horizontal tail plane

$C_{z\text{max}}$: separation!
Four-vortex systems

- two co-cotating pairs

- two counter-cotating pairs

classification chart (Fabre et al. 2002)
Counter-rotating 4-vortex system
(Jacquin et al., ONERA)

- deformations of inner vortices
- large growth rate
- \( \frac{\lambda}{b_1} = O(1) \)
Counter-rotating 4-vortex system

Towing tank experiment
Ortega, Savas (Berkeley)

Numerical simulation
Winckelmanns (UCL)
Short-wave instability
Vortex pairs

Elliptic instability
Short-wave instability ($Re = 2500–4000$, $a/b \approx 0.2$)
Short-wave instability ($Re = 2500–4000, \, a/b \approx 0.2$)
Short-wave instability (Re = 2500–4000, a/b ≈ 0.2)

bottom view
Short-wave instability \( (Re = 2500–4000, \ a/b \approx 0.2) \)
Short-wave instability \((Re = 2500–4000, \ a/b \approx 0.2)\)
Short-wave instability \((Re = 2500–4000, a/b \approx 0.2)\)
Short-wave instability ($Re = 2500–4000$, $a/b \approx 0.2$)
Short-wave instability (spatial structure & symmetry)
Short-wave instability mechanism

*Example:* vortex in a strain - elliptic instability

**Perturbations** of a vortex ("Kelvin modes")

\[ u(r) \cdot \exp[i(kz + m\theta + \omega t)] \]

**Resonance** between two modes \((m_1, k_1, \omega_1)\) and \((m_2, k_2, \omega_2)\) and a strain field is possible if

\[
|m_1 - m_2| = 2 \\
k_1 - k_2 = 0 \\
\omega_1 - \omega_2 = 0
\]

**Amplification** of perturbations

\[ \rightarrow \text{ Instability} \]
Short-wave instability mechanism

*example:* vortex in a strain - elliptic instability

**Experiment**
- Vortex pair

**Theory**
- $m_1=-1$, $m_2=1$

**Experiment**
- Vortex pair with axial flow

**Theory**
- $m_1=2$, $m_2=0$
Short-wave instability in the wake of a Boeing 747
31 March 2004 – flying from Marseille to Frankfurt
(courtesy of Charles Williamson, Cornell)
Short-wave instability in the wake of a Boeing 747
31 March 2004 – flying from Marseille to Frankfurt
(courtesy of Charles Williamson, Cornell)
Long-wave instability
Helical vortices

Pairing instability
Helical vortices
– Applications –

- helicopters
- propellers
- wind turbines

Hand et al. (2001)

Senocak et al. (2002)
Helical vortices
– Applications –

Wind turbines
• Downstream evolution of the rotor wake

Helicopters
• transition from helical wake to Vortex Ring State (VRS)

Ivanell et al. (2010)

Mikkelsen (2010, priv. comm.)

Meijer Drees & Hendal (1950)
Instabilities of helical vortices

- Long-wave displacement instabilities (filaments)
- Short-wave core instabilities (elliptic, curvature)
- Swirling jet instability (vortex breakdown)
Configuration

infinite helical vortices
• circulation $\Gamma$
• radius $R$, pitch $h$ ($\times N$)
• core radius $a$

parameters: $R/a$, $h/a$, $Re=\Gamma/\nu$
$\sim 40$, $\sim 20$, $\sim 10^4$

characteristic scales
• advection velocity : $\sim \Gamma/2h$
• length scale : $h$
• time scale : $2h^2/\Gamma$
$\Rightarrow \sigma^* = \sigma \cdot (2h^2/\Gamma)$
Experimental study

**Water channel:**
- recirculating, free surface
- test section 37 cm × 50 cm × 150 cm
- free stream velocity $U = 35-45$ cm/s

**Rotors:**
- one or two blades
- low-Re airfoil (A18 - M.S. Selig, UIUC)
- tip chord $c = 11$ mm - $Re_c = 30000$
- radius: 80 mm
- constant circulation
- $\Omega = 3-6$ rps tip speed ratio $\lambda = 5-9$

**Methods:**
- dye visualisations
- Particle Image Velocimetry (2C & 3C)
- high-speed video & PIV
Experimental study
Long-wave instability - theoretical results

single helical vortex

\[ \sigma^* = f(\frac{a}{R}, \frac{h}{R}, k) \]

\[ \sigma^* = \sigma \cdot \left(\frac{2h^2}{\Gamma}\right) \]

number of waves in one turn

- \( a/R = 0.1 \)
- \( h/R = \pi/5 = 0.63 \)

- \( k = 1/2 \)
- \( k = 3/2 \)
- \( k = 5/2 \)

\( \sigma^* \) vs wavenumber \( k \)
Long-wave instability - theoretical results

\[ k = \frac{m}{n} \]
\[ m \text{ locations of local pairing} \]
\[ \text{in } n \text{ helix turns} \]
Long-wave instability - theoretical results

(Gupta & Loewy 1974, Okulov & Sørensen 2007)

two interlaced helical vortices

\[ \sigma^* = f(a/R, h/R, k) \]

number of waves in one turn

\[ k = 0 \]

\[ k = 1 \]

\[ k = 2 \]
Perturbed flow - **single** helix

Perturbation of the helix geometry:

\[
\frac{z}{h} = \frac{\theta}{2\pi} + A \cos(k\theta)
\]

*Example:* \( k = 1/2 \) and \( A = 0.15 \) (for \( R/h=2 \))

![Diagram of Widnall mode with \( k = 1/2 \)]

![Graphs showing \( \dot{\theta} \) vs. \( \theta \) and \( t\Omega \)]
Perturbed flow - single helix

$\mathbf{k = 1/2}$ - $\mathbf{A = 0.05}$
Perturbed flow - single helix

\[ k = \frac{3}{2} \quad - \quad A = 0.03 \]

\[ k = \frac{5}{2} \quad - \quad A = 0.01 \]
Perturbed flow - **double** helix

**uniform** pairing
\[ k = 0 \]

**local** pairing
\[ k = 1 \]
Perturbed flow - double helix

**uniform** pairing  
$k = 0$

**local** pairing  
$k = 1$
Growth rate of long-wave instabilities

Experiment vs. theory (Gupta & Loewy 1974)

**single helix**

**two helices**

- **local pairing**, $k \neq 0$, in-phase pert.
- **uniform pairing**, $k = 0$
Pairing instability

2 dimensions (von Kármán & Rubach 1912, Lamb 1932)

infinite row of point vortices

\[ \sigma^* = \frac{\pi}{2} \quad \text{for} \quad \phi = \pi \]

\[ \sigma^* = \sigma \cdot \left( \frac{2b^2}{\Gamma} \right) \]
Pairing instability
2 dimensions (von Kármán & Rubach 1912, Lamb 1932)

infinite row of point vortices

\[ \sigma^* = \sigma \cdot \left( \frac{2b^2}{\Gamma} \right) \]

\[ \sigma^* = \frac{\pi}{2} \text{ for } \phi = \pi \]
Pairing instability

2 dimensions (von Kármán & Rubach 1912, Lamb 1932)

infinite array of straight vortices

\[ \sigma^* = \pi / 2 \quad \text{for} \quad \phi = \pi \]
Pairing instability

3 dimensions (Robinson & Saffman 1982)

infinite array of straight vortices

\[ \lambda / b = 0.1 \]
Pairing instability
prediction for helical geometry

Helical vortex
$R, h, k, \Gamma, a$

Equivalent array of straight vortices
$\mathbf{b} = f(h,R)$
$\phi = f(k,R,h)$
$\lambda = f(k,R,h)$

$\sigma \cdot \left(\frac{2h^2}{\Gamma}\right)$
Long-wave instability - theoretical results

Growth rate for $h/R = \pi/5 = 0.63$

$a/R = 0.1$

Robinson & Saffman (1982)
applied to helical geometry
Long-wave instability - theoretical results

Growth rate for $h/R = \pi/5 = 0.63$

$a/R = 0.2$

Robinson & Saffman (1982) applied to helical geometry
Long-wave instability - theoretical results

Growth rate for $h/R = \pi/5 = 0.63$

$\alpha / R = 0.23$

Robinson & Saffman (1982) applied to helical geometry
Experiments with rotors of more than one blade

Wind turbine in air (Alfredsson & Dahlberg, 1979)

Propeller in water (Felli et al., 2011)
- 2 blades
- 3 blades
- 4 blades

Wind turbine in water (Mikkelsen, 2010)
Application to wind turbines

Huang et al. (2019)

without flap oscillation

with flap oscillation

Clark Y

Vortex Pairing

Leapfrogging
Short-wave instability
Helical vortices

Curvature instability
Short-wave instability mechanism

curved vortex - curvature instability

Kelvin modes

\[ u(r) \cdot \exp[i(kz + m\theta + \omega t)] \]

Resonance between two modes

\((m_1, k_1, \omega_1)\) and \((m_2, k_2, \omega_2)\) and a curvature perturbation if

\[ |m_1 - m_2| = 1 \]
\[ k_1 - k_2 = 0 \]
\[ \omega_1 - \omega_2 = 0 \]

Amplification of perturbations → Instability
Short-wave instability
Theoretical prediction for experimental case
(Blanco-Rodríguez & Le Dizès, 2017)

Gaussian vortex with axial flow

\[ h/R = 0.55 \]
\[ a/R = 0.032 \]
\[ V_z \cdot (2\pi a/\Gamma) = 0.51 \]
\[ Re = 10830 \]

Perturbation vorticity

modes

\[ m_1 = 1, \ m_2 = 0 \]
Axial flow in vortex cores

slow motion: real time / 4
Short-wave instability
Experimental observations

- **wavelength**
  Experiment: $\lambda \approx 6 \, a$
  Theory: $\lambda = 5.7 \, a$

- **mode shape**
  Visualisation compatible with curvature instability

- **growth rate**
  $\sigma^* = 0.15$

slow motion: real time / 8
Short-wave instability
Experimental observations

slow motion: real time / 4
Takeaways

➡ Long-wave instability: vortex displacement, wavelength $\gg$ core size

➡ Short-wave instability: core deformations, wavelength $\approx$ core size

➡ Vortex pairs
  ➥ Long-wave Crow instability
    - Strain + auto-rotation + mutual induction
    - Counter-rotating pair $\rightarrow$ unstable; co-rotating pair $\rightarrow$ stable
  ➥ Short-wave elliptic instability

➡ Helical vortices
  ➥ Long-wave instability
    - Pairing of neighbouring helix loops
  ➥ Short-wave curvature instability
    - Requires axial core flow
End of Lecture 4