Vortex definitions and bifurcations of vortical structures Part 1 — Streamlines and vortex breakdown

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Vorticity, Vortical Flows and Vortex-Induced Vibrations
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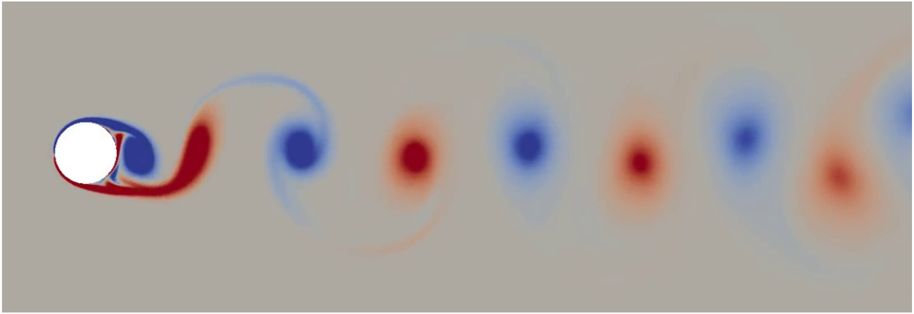
Overview

1. A gallery of vortices
2. What is a vortex?
3. Streamlines, pathlines, streaklines
4. Vortex definition #1
   - Basic topology of 2D incompressible flow
5. Steady vortex breakdown in confined geometry
   - Bifurcation analysis
   - Is the flow really axisymmetric
6. Summary

von Karman vortex street

2S wake

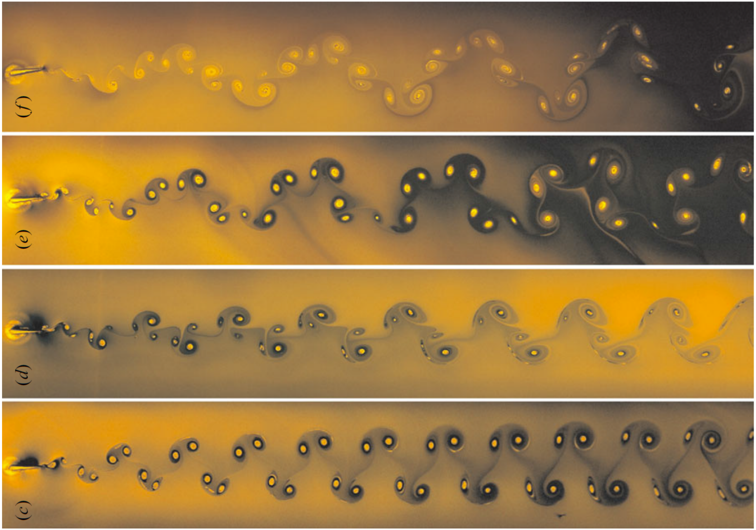


Oscillating cylinder may lead to exotic wakes

P + S wake

Williamson, in Ponta & Aref, J. Fluids Struct. 22(2006), 327–344

Other body shapes produce very exotic wakes



4P + 2S

4P

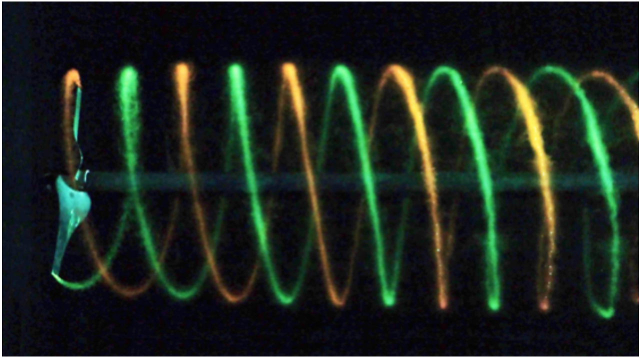
2P + 2S

2P

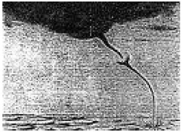
Schnipper *et al*, J. Fluid Mech. 633(2009), 411–423

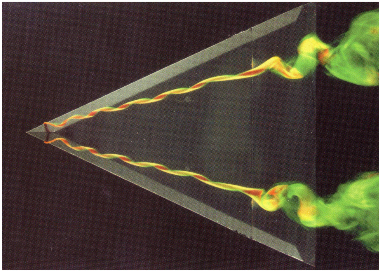
Wingtip vortex

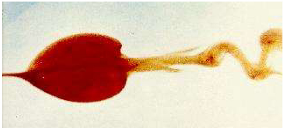
Helical vortex



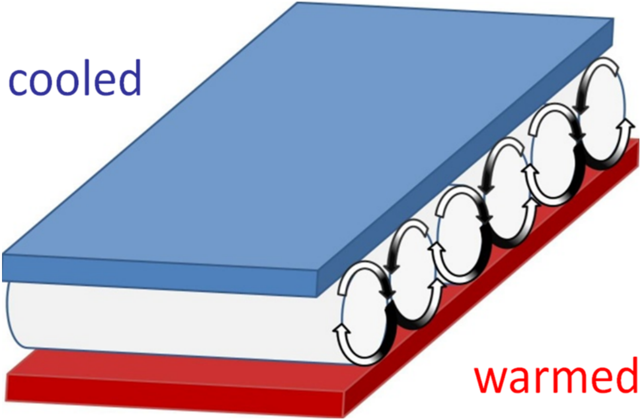
Vortex breakdown



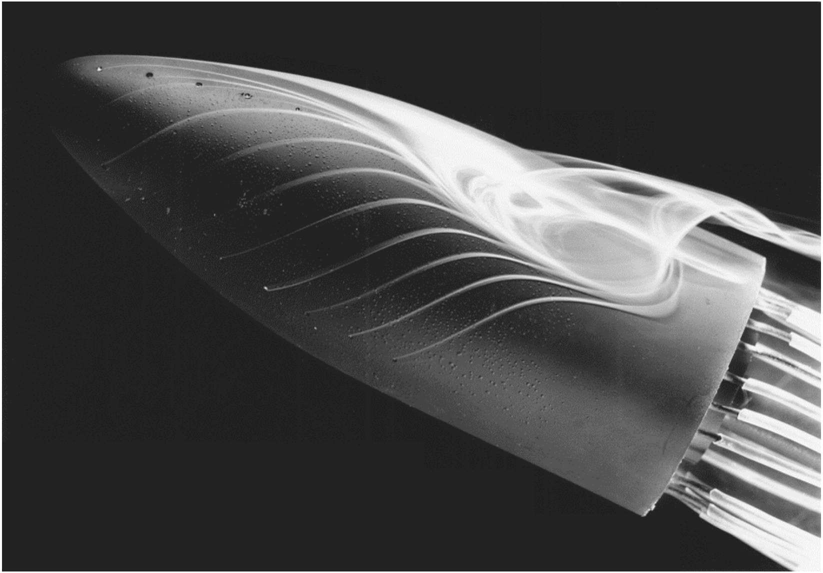




Rayleigh-Bénard convection



3D separation



Overview

What is a vortex?

From Webster's

vortex noun

vor·tex | \ ˈvȯr-ˌteks \

plural **vortices** \ ˈvȯr-tə-ˌsēz \ *also* **vortexes** \ ˈvȯr-ˌtek-səz \

Definition of *vortex*

1 : something that resembles a whirlpool

// the hellish *vortex* of battle

— *Time*

2 a : a mass of fluid (such as a liquid) with a whirling or circular motion that tends to form a cavity or vacuum in the center of the circle and to draw toward this cavity or vacuum bodies subject to its action

especially **:** WHIRLPOOL, EDDY

b : a region within a body of fluid in which the fluid elements have an angular velocity

The notion of a vortex is so widely used in fluid dynamics that few pause to examine what the word strictly means. Those who do take a closer look quickly realize the difficulty of defining vortices unambiguously.

G. Haller, *An objective definition of a vortex*, JFM 2005

Considerable confusion surrounds the longstanding question of what constitutes a vortex, especially in a turbulent flow. This question, frequently misunderstood as academic, has recently acquired particular significance since coherent structures (CS) in turbulent flows are now commonly regarded as vortices

J. Jeong & F. Hussain, *On the identification of a vortex*, JFM 1995

28 definitions of a vortex

Y. Zhang et al. Renewable and Sustainable Energy Reviews 81 (2018) 1269–1285

Table 2
Classification of the existing vortex identification methods in the literature.

Author	Year	Method	Region/Line	Invariant	Local/Global	2D/3D	Objective
Hunt et al. [17]	1988	Q-criterion	Region	Galilean	Local	3D	No
Hunt et al. [17]	1988	Maximum Q	Line	Galilean	Local	3D	No
Chong et al. [18]	1990	$\triangle$-criterion	Region	Galilean	Local	3D	No
Jeong and Hussain [19]	1995	λ_2-criterion	Region	Galilean	Local	3D	No
Jeong and Hussain [19]	1995	Minimum λ_2	Line	Galilean	Local	3D	No
Zhou et al. [20]	1999	Swirling Strength	Region	Galilean	Local	3D	No
Cucitore et al. [21]	1999	Enhanced Swirling Strength	Region	Galilean	Gobal	3D	No
Miliou et al. [22]	2005	Cut-off value λ_2	Region	Galilean	Local	3D	No
Haller [23]	2005	Mz	Region	Lagrangian	Local	3D	Yes
Green et al. [24]	2007	Lyapunov Exponent	Region	Lagrangian	Local	3D	Yes
Fuchs et al. [25]	2008	Delocalized unsteady vortex	Region	Lagrangian	Gobal	3D	No
Gűnther et al. [26]	2016	Rotation invariance	Region	Rotating	Local	3D	No
Berdahl and Thompson [27]	1993	Swirl Parameter	Region	Not	Local	3D	No
Banks and Singer [28]	1995	Predictor-Corrector	Line	Not	Local	3D	No
Cucitore et al. [21]	1999	R-definition	Line	Not	Gobal	3D	No
Lugt [29]	1999	Stream lines	Line	Not	Global	3D	No
Lugt [29]	1999	Path lines	Line	Not	Global	3D	No
Weinkauf and Theisel [30]	2010	Streak lines	line	Not	Global	3D	No
Spalart [31]	1988	Vorticity magnitude	Region	Not	Local	3D	No
Spalart [31]	1988	Vorticity lines	Line	Not	Gocal	3D	No
Kline and Robinson [32]	1990	Pressure iso-surface	Region	Not	Global	3D	No
Kline and Robinson [32]	1990	Pressure minima	Line	Not	Local	2D	No
Degani et al. [33]	1990	Helicity	Line	Not	Local	3D	No
Sujudi and Haimes [34]	1995	Eigenvector	Line	Not	Local	3D	No
Roth and Peikert [35]	1998	Parallel Vectors	Line	Not	Local	3D	No
Holmen [16]	2012	Velocity components	Region	Not	Local	3D	No
Wang and Li [36]	2014	Rotation index-based	Region	Not	Local	2D	No
Dong et al. [37]	2016	Combing λ_2 and vortex filaments	Line	Not	Local	3D	No

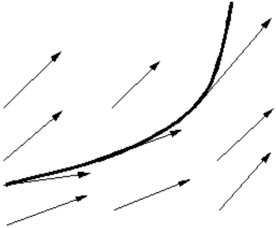
… further definitions appear continuously.

Overview

Streamlines, pathlines, streaklines

For a time-dependent velocity field $\mathbf{v}(\mathbf{x}, t)$:
The *(instantaneous) streamlines at* $t = t_0$ are the solution curves to

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}(\mathbf{x}, t_0)$$



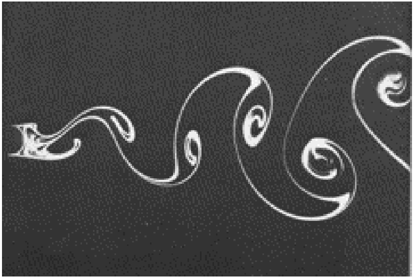
The *pathlines* are the solutions to

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}(\mathbf{x},t).$$

If dye is fed from a point $\mathbf{x}_0$ a *streakline* appears in the flow. if the pathline which fulfills the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$ is denoted $\mathbf{x}(t_0,t)$ the streakline at time t is the curve

$$t_0 \to \mathbf{x}(t_0,t), \quad t_0 \in [t_s,t],$$

where t_s is the time the experiment (or dye release) is started.



- If the flow is steady, $\mathbf{v}(\mathbf{x},t) = \mathbf{v}(\mathbf{x})$, streamlines, pathlines and streaklines coincide.
- In two-dimensional incompressible flow there is a *streamfunction* $\psi(\mathbf{x},t)$ such that

$$\mathbf{v} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \nabla\psi = \begin{pmatrix} \frac{\partial\psi}{\partial y} \\ -\frac{\partial\psi}{\partial x} \end{pmatrix}.$$

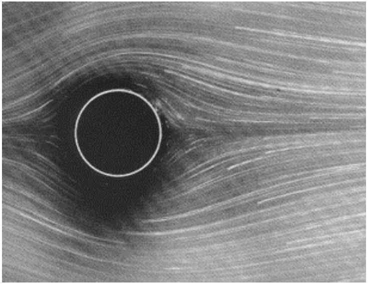
  The streamlines are the level curves of ψ.

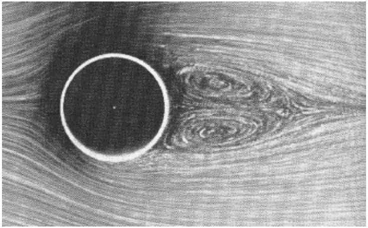
Overview

Vortex defintion # 1

A vortex in a 2D flow is a region with closed streamlines.

$Re = 1.54$ $Re = 26$





Not Galilean invariant — two observers moving with a constant relative speed will not find the same streamline structure.

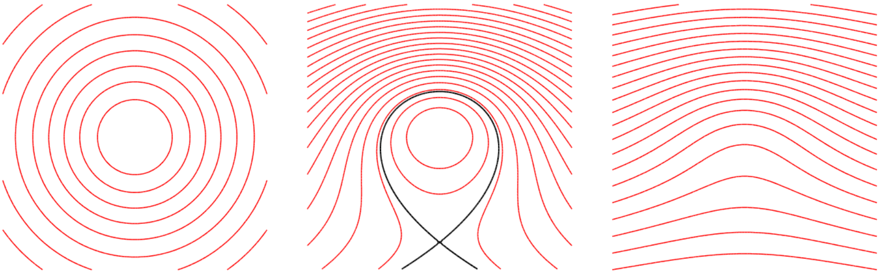
Gaussian vortex in a background flow $\mathbf{U}$

Vorticity and streamfunction for a Gaussian vortex

$$\omega = e^{-r^2}, \qquad \psi = -\frac{1}{4}\left(\ln(r^2) + \int_1^\infty \frac{e^{-ar^2}}{a}\,da\right), \qquad r^2 = x^2 + y^2$$

Streamlines

$\mathbf{U} = \mathbf{0}$ $\qquad$ $\mathbf{U} = \begin{pmatrix} -0.2 \\ 0 \end{pmatrix}$ $\qquad$ $\mathbf{U} = \begin{pmatrix} -0.4 \\ 0 \end{pmatrix}$

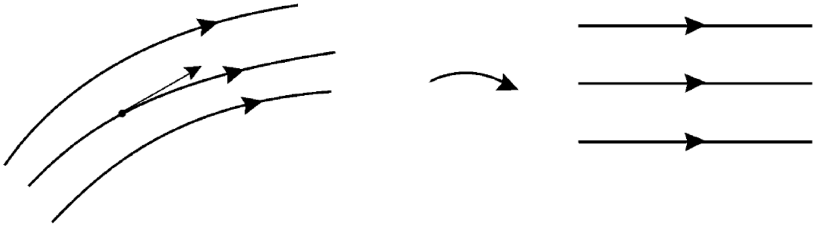


Streamline structure depends on the velocity of the observer – only meaningful where there is a distinguished coordinate system such as in steady flows.

Basic topology of 2D incompressible flow

A *regular point* is a point where the velocity is different from zero.

The flow box theorem In a neighborhood of a regular point the flow field can be transformed into a constant velocity field.



Interesting behavior occurs near *critical points* where the velocity is zero, $v_x = v_y = 0$, that is

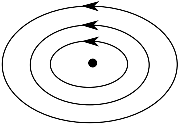
$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} = 0.$$

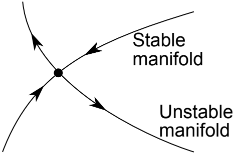
Types of critical points

The Hessian matrix

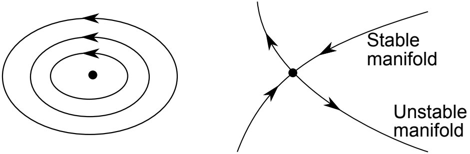
$$H = \begin{pmatrix} \frac{\partial^2\psi}{\partial x^2} & \frac{\partial^2\psi}{\partial x \partial y} \\ \frac{\partial^2\psi}{\partial y \partial x} & \frac{\partial^2\psi}{\partial y^2} \end{pmatrix}$$

evaluated at a critical point determines its type: If the eigenvalues have the same sign, the point is a local extremum (center), if the eigenvalues have opposite signs, it is a *saddle*.





Degenerate critical points

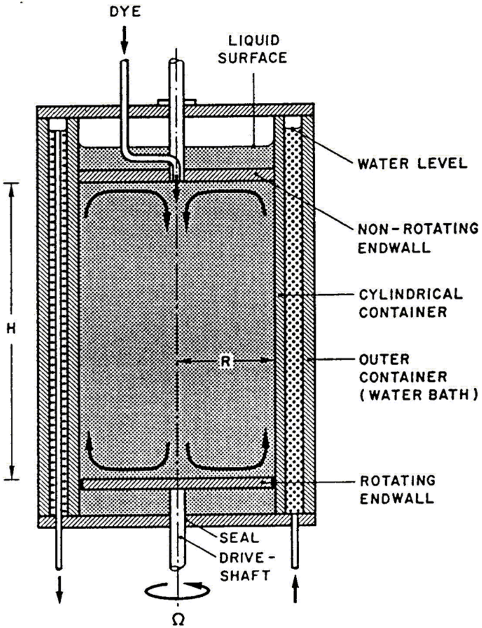


- Centers and saddles are *structurally stable*, that is, they persist under perturbations. Topological bifurcation is impossible.
- If zero is an eigenvalue of the Hessian, the critical point is *degenerate* and *structurally unstable*. Topological bifurcation is possible.

We will explore topological bifurcations in the context of secondary structures on a main vortex in axisymmetric flow.

Overview

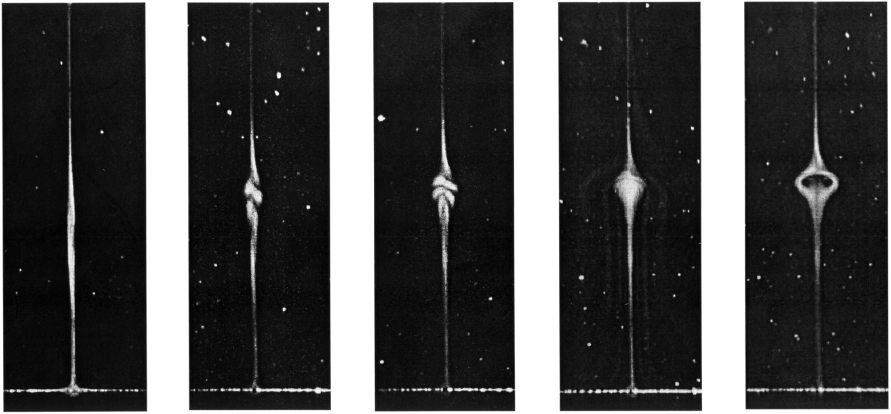
Steady vortex breakdown in confined geometry



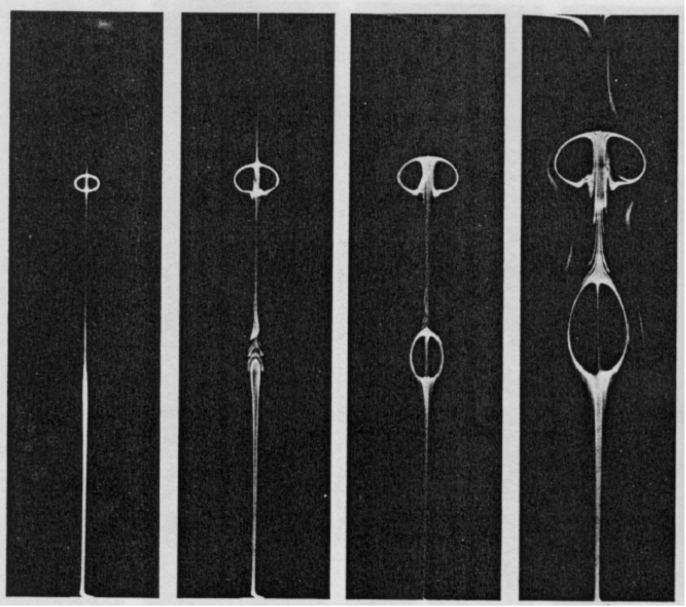
- Also with co- or counter rotating lid or free surface
- Of independent interest as a bioreactor
- Two dimensionless parameters

$$h = \frac{H}{R}, \quad Re = \frac{\Omega R^2}{\nu}$$

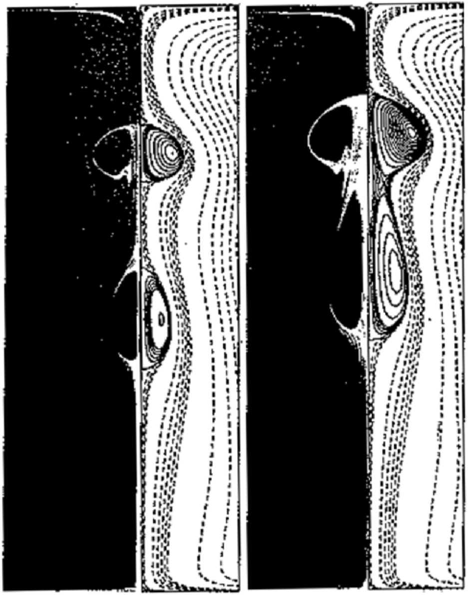
Development of recirculation zone as *Re* is increased



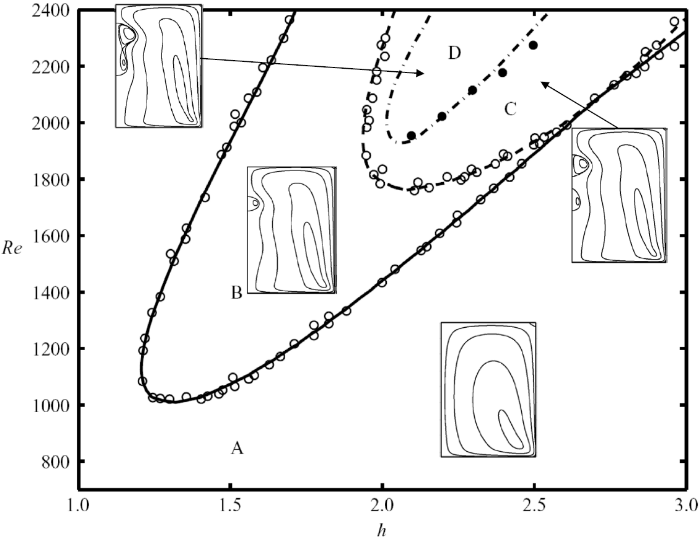
Multiple breakdowns possible



The flow is well described assuming axisymmetry



Topological bifurcation diagram in the steady regime



Stream function for axisymmetric flows

Equation of continuity in cylindrical coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta) + \frac{\partial}{\partial z}(v_z) = 0.$$

Axisymmetric flow

$$\frac{\partial}{\partial r}(rv_r) + \frac{\partial}{\partial z}(rv_z) = 0.$$

Hence there exists a function ψ such that

$$\frac{\partial \psi}{\partial z} = rv_r, \quad -\frac{\partial \psi}{\partial r} = rv_z.$$

New radial variable $x = r^2/2$

$$v_x = rv_r = \frac{\partial \psi}{\partial z}, \quad v_z = -\frac{\partial \psi}{\partial x}.$$

Intersection of stream-surfaces with a plane $\theta = \theta_0$ are level curves of ψ. Consider bifurcations of level curves of ψ as creation (or destruction) of secondary vortices on the main vortex axis

Bifurcation analysis

Step #1: Expand the stream function in a Taylor series around a point on the axis chosen to be the origo,

$$\psi = x(a_{10} + a_{11}z + a_{20}x + a_{30}x^2 + a_{21}xz + a_{12}z^2 + \cdots).$$

The factor x ensures that the axis $x = 0$ is a streamline.

$$\frac{\partial\psi}{\partial x} = a_{10} + a_{11}z + 2a_{20}x + 3a_{30}x^2 + 2a_{21}xz + a_{12}z^2 + \cdots,$$

$$\frac{\partial\psi}{\partial z} = x(a_{11} + a_{21}x + 2a_{12}z + \cdots).$$

The origin is a critical point if $a_{10} = 0$. The Hessian is

$$H = \begin{pmatrix} 2a_{20} & a_{11} \\ a_{11} & 0 \end{pmatrix}.$$

Zero is an eigenvalue if $a_{11} = 0$. Assume we vary a_{10}, a_{11} away from their degenerate values, relabeling them

$$a_{10} = \epsilon_1, \quad a_{11} = \epsilon_2,$$

to mark them as perturbation parameters.

$$\psi = x(\epsilon_1 + \epsilon_2 z + a_{20}x + a_{30}x^2 + a_{21}xz + a_{12}z^2 + \cdots).$$

Step # 2: Use a series of coordinate transformations to simplify ψ as much as possible. First a *near-identity transformation* preserving the axis,

$$x = \xi + s_{20}\xi^2 + s_{11}\xi\eta, \quad z = \eta.$$

This yields

$$\begin{aligned}\psi = {} & \xi(\epsilon_1 + (s_{20}\epsilon_1 + \epsilon_2)\xi + (s_{11}\epsilon_1 + \epsilon_2)\eta \\ & + (2s_{20}a_{20} + a_{30})\xi^2 + (s_{20}\epsilon_1 + 2a_{20}s_{11} + a_{21})\xi\eta + (s_{11}\epsilon_2 + a_{12})\eta^2 + \cdots).\end{aligned}$$

Choose s_{20}, s_{11} such that the $\xi^2\eta, \xi^3$ terms vanish,

$$s_{20} = -\frac{a_{30}}{2a_{20}}, \quad s_{11} = \frac{a_{30}\epsilon_2 - 2a_{20}a_{21}}{4a_{20}^2}.$$

Must assume $a_{20} \neq 0$.

$$\psi = \xi \left(\epsilon_1 + \frac{2a_{20}^2 - a_{30}\epsilon_1}{2a_{20}}\xi + \frac{a_{30}\epsilon_1\epsilon_2 - 2a_{20}a_{21}\epsilon_1 + 4a_{20}^2\epsilon_2}{4a_{20}^2}\eta \right.$$
$$\left. + \frac{a_{30}\epsilon_2^2 - 2a_{20}a_{21}\epsilon_2 + 4a_{12}a_{20}^2}{4a_{20}^2}\eta^2 + \cdots \right)$$

Step #3: Translate along the η-axis, $\eta \to \eta + \eta_0$, to remove the $\xi\eta$ term. This requires $a_{12} \neq 0$.
Step # 4: Scale the variables and ψ and rename the variables back to (x,z) to obtain

$$\psi = x(\mu + \sigma x + z^2 + \cdots)$$

where μ is a transformed small parameter including ϵ_1, ϵ_2,

$$\mu = -\frac{(a_{20}a_{21} - a_{30}\epsilon_2)^2\epsilon_1^2 - 8a_{20}^2(a_{30}\epsilon_2^2 - 2a_{20}a_{21}\epsilon_2 + 8a_{12}a_{20}^2)\epsilon_1 + 16a_{20}^4\epsilon_2^2}{4(a_{12}a_{20}^2 - 2a_{20}a_{21}\epsilon_2 + a_{30}\epsilon_2^2)},$$

and $\sigma = \mathrm{sign}(a_{02}a_{12})$.

Summary: Assuming a_{10},a_{11} small parameters and a_{02},$a_{12} \neq 0$, the stream function

$$\psi = x(\epsilon_1 + \epsilon_2 z + a_{20}x + a_{30}x^2 + a_{21}xz + a_{12}z^2 + \cdots)$$

is transformed into

$$\psi = x(\mu + \sigma x + z^2 + \cdots), \quad \mu \text{ small }, \sigma = \pm 1$$

The truncated normal form is obtained by dropping the higher-order terms,

$$\psi = x(\mu + \sigma x + z^2)$$

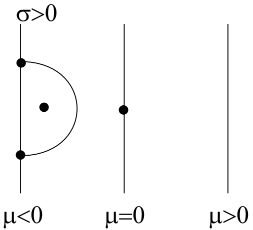
Analyzing the truncated normal form

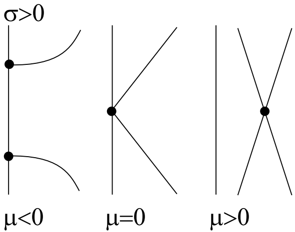
$$\psi = x(\mu + \sigma x + z^2), \quad \frac{\partial \psi}{\partial x} = \mu + 2\sigma x + z^2, \qquad \frac{\partial \psi}{\partial z} = 2xz.$$

Critical points: $\partial\psi/\partial z = 0$ for $x = 0$ or $z = 0$.

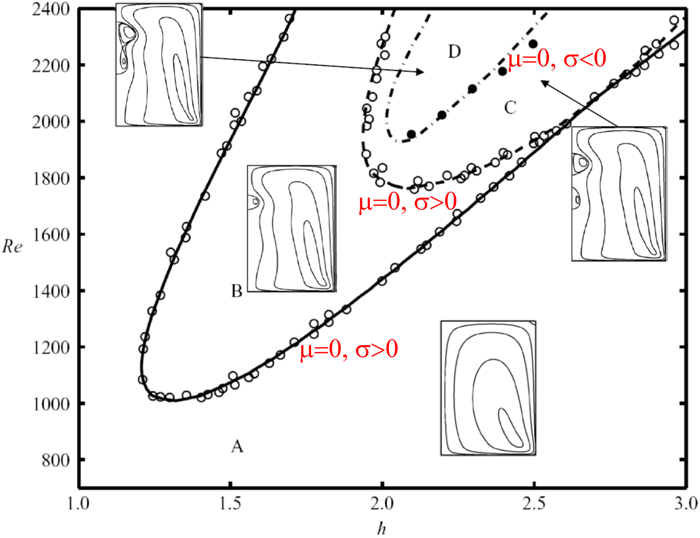
For $x = 0$, $\partial\psi/\partial x = 0$ when $z = \pm\sqrt{-\mu}$.

For $z = 0$, $v_z = 0$ when $x = -\sigma\mu/2$.

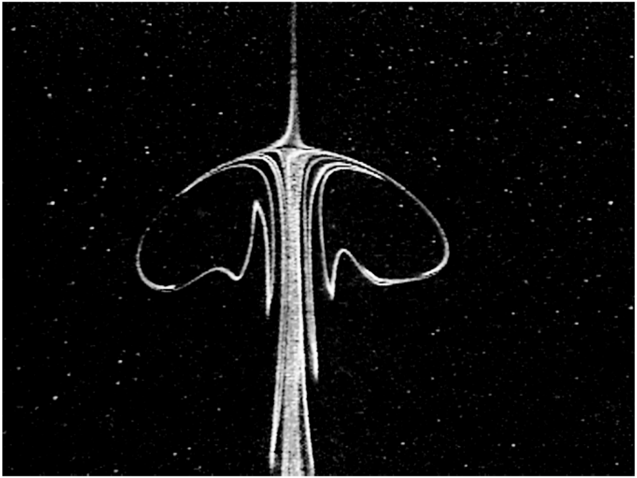




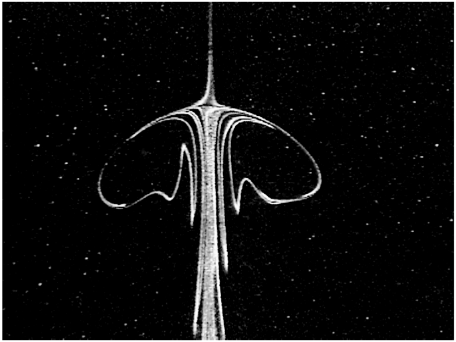
Vortex breakdown bifurcation diagram again



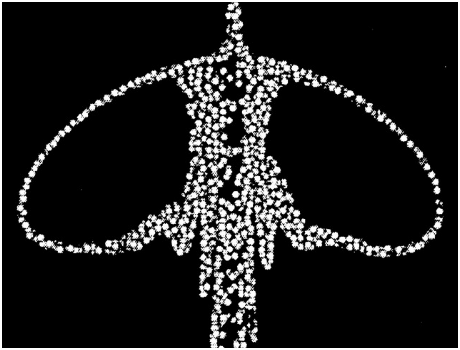
Is the flow really axisymmetric?



Electrostatic precipitation visualization by Spohn (1998)



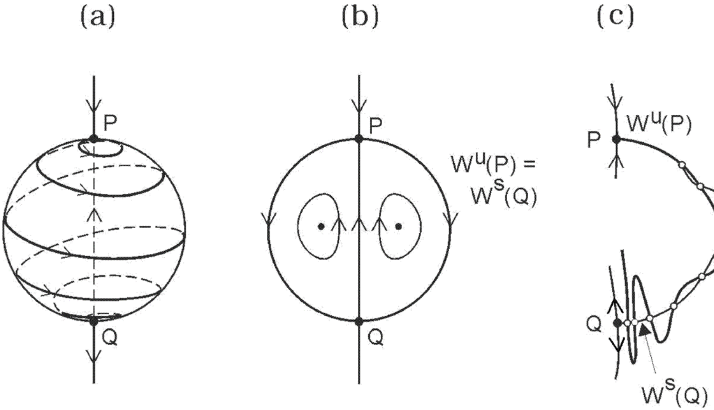
3D computations by Sotiropoulos and Ventikos (2001)



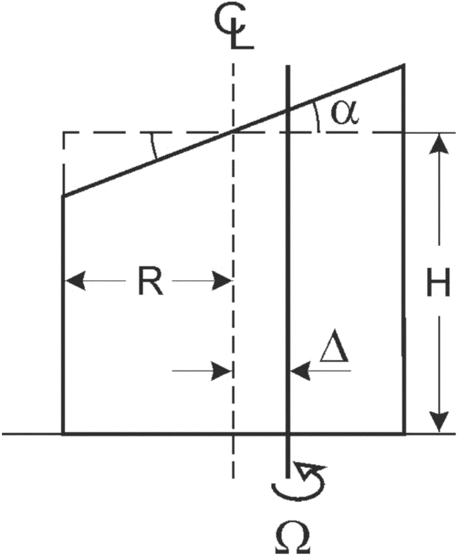
3D stability analysis by Gelfgat et al. (2001): The axisymmetric flow is stable with respect to all perturbation in the relevant parameter regime

Overall flow structure is very well predicted by axisymmetric simulations

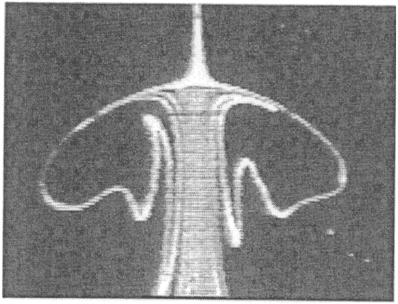
Structural instability of the axisymmetric breakdown bubble



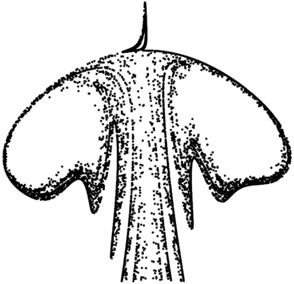
The role of geometric imperfections



Axisymmetric + experimental error

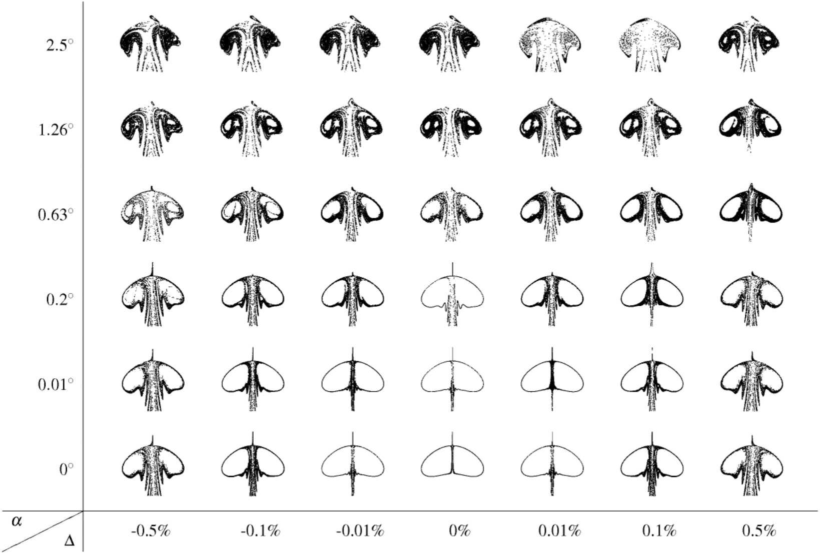


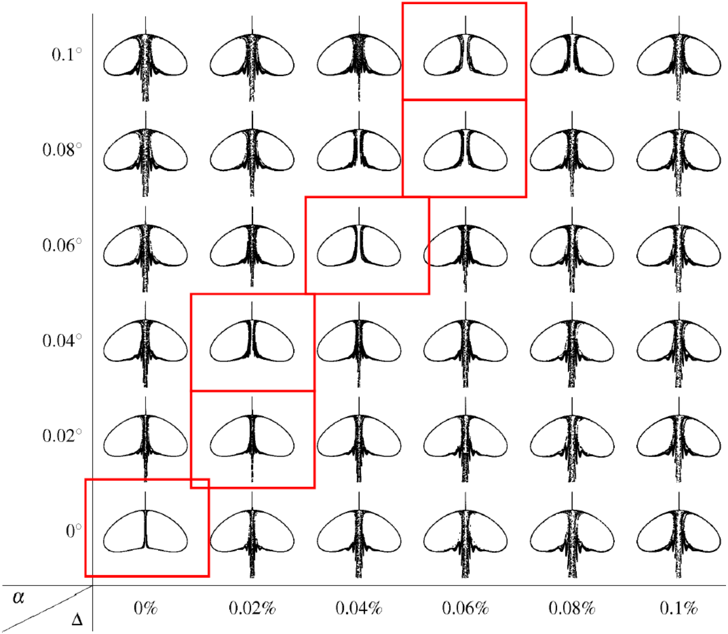
$\alpha = 0, \Delta = -0.5\%$



$\Delta = -0.5\%, R = 45.7\text{mm}$: Displacement of axis is 0.23mm

Imperfections may annihilate




0.1°
0.08°
0.06°
0.04°
0.02°
0°
α
Δ
0%
0.02%
0.04%
0.06%
0.08%
0.1%

Overview

Summary

- There is no universally accepted mathematical definition of a vortex
- Defining vortices on the basis of streamlines makes sense for steady flows
- Bifurcations of streamline patterns may be studied with the aid of normal form transformations
- The lack of structural stability of axisymmetric flows gives rise to remarkable asymmetric effects